

# CONTENTS

<i>List of Worked Examples</i>	<i>page</i> xix
<i>Preface</i>	xxi
<i>Two Paths Through the Book</i>	xxiv
<b>1 FUNDAMENTALS OF MECHANICS</b>	<b>1</b>
1.1 Elementary Kinematics	1
1.1.1 Trajectories of Point Particles	1
1.1.2 Position, Velocity, and Acceleration	3
1.2 Principles of Dynamics	5
1.2.1 Newton's Laws	5
1.2.2 The Two Principles	6
Principle 1	7
Principle 2	7
Discussion	9
1.2.3 Consequences of Newton's Equations	10
Introduction	10
Force is a Vector	11
1.3 One-Particle Dynamical Variables	13
1.3.1 Momentum	14
1.3.2 Angular Momentum	14
1.3.3 Energy and Work	15
In Three Dimensions	15
Application to One-Dimensional Motion	18
1.4 Many-Particle Systems	22
1.4.1 Momentum and Center of Mass	22
Center of Mass	22
Momentum	24
Variable Mass	24
1.4.2 Energy	26
1.4.3 Angular Momentum	27

1.5 Examples	29
1.5.1 Velocity Phase Space and Phase Portraits	29
The Cosine Potential	29
The Kepler Problem	31
1.5.2 A System with Energy Loss	34
1.5.3 Noninertial Frames and the Equivalence Principle	38
Equivalence Principle	38
Rotating Frames	41
Problems	42
<b>2 LAGRANGIAN FORMULATION OF MECHANICS</b>	48
2.1 Constraints and Configuration Manifolds	49
2.1.1 Constraints	49
Constraint Equations	49
Constraints and Work	50
2.1.2 Generalized Coordinates	54
2.1.3 Examples of Configuration Manifolds	57
The Finite Line	57
The Circle	57
The Plane	57
The Two-Sphere $S^2$	57
The Double Pendulum	60
Discussion	60
2.2 Lagrange's Equations	62
2.2.1 Derivation of Lagrange's Equations	62
2.2.2 Transformations of Lagrangians	67
Equivalent Lagrangians	67
Coordinate Independence	68
Hessian Condition	69
2.2.3 Conservation of Energy	70
2.2.4 Charged Particle in an Electromagnetic Field	72
The Lagrangian	72
A Time-Dependent Coordinate Transformation	74
2.3 Central Force Motion	77
2.3.1 The General Central Force Problem	77
Statement of the Problem; Reduced Mass	77
Reduction to Two Freedoms	78
The Equivalent One-Dimensional Problem	79
2.3.2 The Kepler Problem	84
2.3.3 Bertrand's Theorem	88
2.4 The Tangent Bundle $T\mathbb{Q}$	92

2.4.1 Dynamics on $T\mathbb{Q}$	92
Velocities Do Not Lie in $\mathbb{Q}$	92
Tangent Spaces and the Tangent Bundle	93
Lagrange's Equations and Trajectories on $T\mathbb{Q}$	95
2.4.2 $T\mathbb{Q}$ as a Differential Manifold	97
Differential Manifolds	97
Tangent Spaces and Tangent Bundles	100
Application to Lagrange's Equations	102
Problems	103
<b>3 TOPICS IN LAGRANGIAN DYNAMICS</b>	108
3.1 The Variational Principle and Lagrange's Equations	108
3.1.1 Derivation	108
The Action	108
Hamilton's Principle	110
Discussion	112
3.1.2 Inclusion of Constraints	114
3.2 Symmetry and Conservation	118
3.2.1 Cyclic Coordinates	118
Invariant Submanifolds and Conservation of Momentum	118
Transformations, Passive and Active	119
Three Examples	123
3.2.2 Noether's Theorem	124
Point Transformations	124
The Theorem	125
3.3 Nonpotential Forces	128
3.3.1 Dissipative Forces in the Lagrangian Formalism	129
Rewriting the EL Equations	129
The Dissipative and Rayleigh Functions	129
3.3.2 The Damped Harmonic Oscillator	131
3.3.3 Comment on Time-Dependent Forces	134
3.4 A Digression on Geometry	134
3.4.1 Some Geometry	134
Vector Fields	134
One-Forms	135
The Lie Derivative	136
3.4.2 The Euler–Lagrange Equations	138
3.4.3 Noether's Theorem	139
One-Parameter Groups	139
The Theorem	140
Problems	143

<b>4 SCATTERING AND LINEAR OSCILLATIONS</b>	147
4.1 Scattering	147
4.1.1 Scattering by Central Forces	147
General Considerations	147
The Rutherford Cross Section	153
4.1.2 The Inverse Scattering Problem	154
General Treatment	154
Example: Coulomb Scattering	156
4.1.3 Chaotic Scattering, Cantor Sets, and Fractal Dimension	157
Two Disks	158
Three Disks, Cantor Sets	162
Fractal Dimension and Lyapunov Exponent	166
Some Further Results	169
4.1.4 Scattering of a Charge by a Magnetic Dipole	170
The Störmer Problem	170
The Equatorial Limit	171
The General Case	174
4.2 Linear Oscillations	178
4.2.1 Linear Approximation: Small Vibrations	178
Linearization	178
Normal Modes	180
4.2.2 Commensurate and Incommensurate Frequencies	183
The Invariant Torus $\mathbb{T}$	183
The Poincaré Map	185
4.2.3 A Chain of Coupled Oscillators	187
General Solution	187
The Finite Chain	189
4.2.4 Forced and Damped Oscillators	192
Forced Undamped Oscillator	192
Forced Damped Oscillator	193
Problems	197
<b>5 HAMILTONIAN FORMULATION OF MECHANICS</b>	201
5.1 Hamilton's Canonical Equations	202
5.1.1 Local Considerations	202
From the Lagrangian to the Hamiltonian	202
A Brief Review of Special Relativity	207
The Relativistic Kepler Problem	211
5.1.2 The Legendre Transform	212

5.1.3 Unified Coordinates on $T^*Q$ and Poisson Brackets	215
The $\xi$ Notation	215
Variational Derivation of Hamilton's Equations	217
Poisson Brackets	218
Poisson Brackets and Hamiltonian Dynamics	222
5.2 Symplectic Geometry	224
5.2.1 The Cotangent Manifold	224
5.2.2 Two-Forms	225
5.2.3 The Symplectic Form $\omega$	226
5.3 Canonical Transformations	231
5.3.1 Local Considerations	231
Reduction on $T^*Q$ by Constants of the Motion	231
Definition of Canonical Transformations	232
Changes Induced by Canonical Transformations	234
Two Examples	236
5.3.2 Intrinsic Approach	239
5.3.3 Generating Functions of Canonical Transformations	240
Generating Functions	240
The Generating Functions Gives the New Hamiltonian	242
Generating Functions of Type	244
5.3.4 One-Parameter Groups of Canonical Transformations	248
Infinitesimal Generators of One-Parameter Groups; Hamiltonian Flows	249
The Hamiltonian Noether Theorem	251
Flows and Poisson Brackets	252
5.4 Two Theorems: Liouville and Darboux	253
5.4.1 Liouville's Volume Theorem	253
Volume	253
Integration on $T^*Q$ ; The Liouville Theorem	257
Poincaré Invariants	260
Density of States	261
5.4.2 Darboux's Theorem	268
The Theorem	269
Reduction	270
Problems	275
Canonicity Implies PB Preservation	280

<b>6 TOPICS IN HAMILTONIAN DYNAMICS</b>	
6.1 The Hamilton–Jacobi Method	284
6.1.1 The Hamilton–Jacobi Equation	284
Derivation	285
Properties of Solutions	286
Relation to the Action	288
6.1.2 Separation of Variables	290
The Method of Separation	291
Example: Charged Particle in a Magnetic Field	294
6.1.3 Geometry and the HJ Equation	301
6.1.4 The Analogy Between Optics and the HJ Method	303
6.2 Completely Integrable Systems	307
6.2.1 Action–Angle Variables	307
Invariant Tori	307
The $\phi^\alpha$ and $J_\alpha$	309
The Canonical Transformation to AA Variables	311
Example: A Particle on a Vertical Cylinder	314
6.2.2 Liouville’s Integrability Theorem	320
Complete Integrability	320
The Tori	321
The $J_\alpha$	323
Example: the Neumann Problem	324
6.2.3 Motion on the Tori	328
Rational and Irrational Winding Lines	328
Fourier Series	331
6.3 Perturbation Theory	332
6.3.1 Example: The Quartic Oscillator; Secular Perturbation Theory	332
6.3.2 Hamiltonian Perturbation Theory	336
Perturbation via Canonical Transformations	337
Averaging	339
Canonical Perturbation Theory in One Freedom	340
Canonical Perturbation Theory in Many Freedoms	346
The Lie Transformation Method	351
Example: The Quartic Oscillator	357
6.4 Adiabatic Invariance	359
6.4.1 The Adiabatic Theorem	360
Oscillator with Time-Dependent Frequency	360
The Theorem	361
Remarks on $N > 1$	363
6.4.2 Higher Approximations	364

<b>6.4.3 The Hannay Angle</b>	365
6.4.4 Motion of a Charged Particle in a Magnetic Field	371
The Action Integral	371
Three Magnetic Adiabatic Invariants	374
Problems	377
<b>7 NONLINEAR DYNAMICS</b>	382
7.1 Nonlinear Oscillators	383
7.1.1 A Model System	383
7.1.2 Driven Quartic Oscillator	386
Damped Driven Quartic Oscillator; Harmonic Analysis	387
Undamped Driven Quartic Oscillator	390
7.1.3 Example: The van der Pol Oscillator	391
7.2 Stability of Solutions	396
7.2.1 Stability of Autonomous Systems	397
Definitions	397
The Poincaré–Bendixon Theorem	399
Linearization	400
7.2.2 Stability of Nonautonomous Systems	410
The Poincaré Map	410
Linearization of Discrete Maps	413
Example: The Linearized Hénon Map	417
7.3 Parametric Oscillators	418
7.3.1 Floquet Theory	419
The Floquet Operator $\mathbf{R}$	419
Standard Basis	420
Eigenvalues of $\mathbf{R}$ and Stability	421
Dependence on $G$	424
7.3.2 The Vertically Driven Pendulum	424
The Mathieu Equation	424
Stability of the Pendulum	426
The Inverted Pendulum	427
Damping	429
7.4 Discrete Maps; Chaos	431
7.4.1 The Logistic Map	431
Definition	432
Fixed Points	432
Period Doubling	434
Universality	442
Further Remarks	444

7.4.2 The Circle Map	445
The Damped Driven Pendulum	445
The Standard Sine Circle Map	446
Rotation Number and the Devil's Staircase	447
Fixed Points of the Circle Map	450
7.5 Chaos in Hamiltonian Systems and the KAM Theorem	452
7.5.1 The Kicked Rotator	453
The Dynamical System	453
The Standard Map	454
Poincaré Map of the Perturbed System	455
7.5.2 The Hénon Map	460
7.5.3 Chaos in Hamiltonian Systems	463
Poincaré–Birkhoff Theorem	464
The Twist Map	466
Numbers and Properties of the Fixed Points	467
The Homoclinic Tangle	468
The Transition to Chaos	472
7.5.4 The KAM Theorem	474
Background	474
Two Conditions: Hessian and Diophantine	475
The Theorem	477
A Brief Description of the Proof of KAM	480
Problems	483
Number Theory	486
The Unit Interval	486
A Diophantine Condition	487
The Circle and the Plane	488
KAM and Continued Fractions	489
<b>8 RIGID BODIES</b>	492
8.1 Introduction	492
8.1.1 Rigidity and Kinematics	492
Definition	492
The Angular Velocity Vector $\omega$	493
8.1.2 Kinetic Energy and Angular Momentum	495
Kinetic Energy	495
Angular Momentum	498
8.1.3 Dynamics	499
Space and Body Systems	499
Dynamical Equations	500
Example: The Gyrocompass	503

Motion of the Angular Momentum $\mathbf{J}$	505
Fixed Points and Stability	506
The Poinsot Construction	508
8.2 The Lagrangian and Hamiltonian Formulations	510
8.2.1 The Configuration Manifold $\mathbb{Q}_R$	510
Inertial, Space, and Body Systems	510
The Dimension of $\mathbb{Q}_R$	511
The Structure of $\mathbb{Q}_R$	512
8.2.2 The Lagrangian	514
Kinetic Energy	514
The Constraints	515
8.2.3 The Euler–Lagrange Equations	516
Derivation	516
The Angular Velocity Matrix $\Omega$	518
8.2.4 The Hamiltonian Formalism	519
8.2.5 Equivalence to Euler's Equations	520
Antisymmetric Matrix–Vector Correspondence	520
The Torque	521
The Angular Velocity Pseudovector and Kinematics	522
Transformations of Velocities	523
Hamilton's Canonical Equations	524
8.2.6 Discussion	525
8.3 Euler Angles and Spinning Tops	526
8.3.1 Euler Angles	526
Definition	526
$R$ in Terms of the Euler Angles	527
Angular Velocities	529
Discussion	531
8.3.2 Geometric Phase for a Rigid Body	533
8.3.3 Spinning Tops	535
The Lagrangian and Hamiltonian	536
The Motion of the Top	537
Nutation and Precession	539
Quadratic Potential; the Neumann Problem	542
8.4 Cayley–Klein Parameters	543
8.4.1 $2 \times 2$ Matrix Representation of 3-Vectors and Rotations	543
3-Vectors	543
Rotations	544
8.4.2 The Pauli Matrices and CK Parameters	544
Definitions	544

Finding $R_U$	545
Axis and Angle in terms of the CK Parameters	546
8.4.3 Relation Between SU(2) and SO(3)	547
Problems	549
<b>9 CONTINUUM DYNAMICS</b>	553
9.1 Lagrangian Formulation of Continuum Dynamics	553
9.1.1 Passing to the Continuum Limit	553
The Sine–Gordon Equation	553
The Wave and Klein–Gordon Equations	556
9.1.2 The Variational Principle	557
Introduction	557
Variational Derivation of the EL Equations	557
The Functional Derivative	560
Discussion	560
9.1.3 Maxwell’s Equations	561
Some Special Relativity	561
Electromagnetic Fields	562
The Lagrangian and the EL Equations	564
9.2 Noether’s Theorem and Relativistic Fields	565
9.2.1 Noether’s Theorem	565
The Theorem	565
Conserved Currents	566
Energy and Momentum in the Field	566
Example: The Electromagnetic Energy–Momentum Tensor	567
9.2.2 Relativistic Fields	569
Lorentz Transformations	571
Lorentz Invariant $\mathcal{L}$ and Conservation	571
Free Klein–Gordon Fields	572
Complex K–G Field and Interaction with the Maxwell Field	576
Discussion of the Coupled Field Equations	577
9.2.3 Spinors	579
Spinor Fields	580
A Spinor Field Equation	582
9.3 The Hamiltonian Formalism	583
9.3.1 The Hamiltonian Formalism for Fields	583
Definitions	583
The Canonical Equations	584
Poisson Brackets	586

9.3.2 Expansion in Orthonormal Functions	588
Orthonormal Functions	589
Particle-like Equations	590
Example: Klein–Gordon	591
9.4 Nonlinear Field Theory	594
9.4.1 The Sine–Gordon Equation	594
Soliton Solutions	595
Properties of SG Solitons	597
Multiple-Soliton Solutions	599
Generating Soliton Solutions	601
Nonsoliton Solutions	605
Josephson Junctions	608
9.4.2 The Nonlinear K–G Equation	608
The Lagrangian and the EL Equation	608
Kinks	609
9.5 Fluid Dynamics	610
9.5.1 The Euler and Navier–Stokes Equations	611
Substantial Derivative and Mass Conservation	611
Euler’s Equation	612
Viscosity and Incompressibility	614
The Navier–Stokes Equations	615
Turbulence	616
9.5.2 The Burgers Equation	618
The Equation	618
Asymptotic Solution	620
9.5.3 Surface Waves	622
Equations for the Waves	622
Linear Gravity Waves	624
Nonlinear Shallow Water Waves: the KdV Equation	626
Single KdV Solitons	629
Multiple KdV Solitons	631
9.6 Hamiltonian Formalism for Nonlinear Field Theory	632
9.6.1 The Field Theory Analog of Particle Dynamics	633
From Particles to Fields	633
Dynamical Variables and Equations of Motion	634
9.6.2 The Hamiltonian Formalism	634
The Gradient	634
The Symplectic Form	636
The Condition for Canonicity	636
Poisson Brackets	636
9.6.3 The kdV Equation	637
KdV as a Hamiltonian Field	637

Constants of the Motion	638
Generating the Constants of the Motion	639
More on Constants of the Motion	640
9.6.4. The Sine–Gordon Equation	642
Two-Component Field Variables	642
$sG$ as a Hamiltonian Field	643
Problems	646
<b>EPILOGUE</b>	648
<b>APPENDIX: VECTOR SPACES</b>	649
General Vector Spaces	649
Linear Operators	651
Inverses and Eigenvalues	652
Inner Products and Hermitian Operators	653
<b>BIBLIOGRAPHY</b>	656
<b>INDEX</b>	663