
Contents

1	Introduction	1
1.1	Quantification of physical phenomena	1
1.1.1	Observation of physical phenomena	1
1.1.2	Mathematical model	2
1.1.3	Numerical model	2
1.1.4	Assessment of the numerical results	2
1.2	Linear and nonlinear mathematical models	2
1.3	The aims of this book	4
1.4	Notation	5
2	Kinematics of the continuous media	7
2.1	The continuous media and its configurations	7
2.2	Mass of the continuous media	9
2.3	Motion of continuous bodies	9
2.3.1	Displacements	9
2.3.2	Velocities and accelerations	10
2.4	Material and spatial derivatives of a tensor field	12
2.5	Convected coordinates	13
2.6	The deformation gradient tensor	13
2.7	The polar decomposition	21
2.7.1	The Green deformation tensor	21
2.7.2	The right polar decomposition	22
2.7.3	The Finger deformation tensor	25
2.7.4	The left polar decomposition	25
2.7.5	Physical interpretation of the tensors $\overset{t}{\underline{\mathbf{R}}}$, $\overset{t}{\underline{\mathbf{U}}}$ and $\overset{t}{\underline{\mathbf{V}}}$	26
2.7.6	Numerical algorithm for the polar decomposition	28
2.8	Strain measures	33
2.8.1	The Green deformation tensor	33
2.8.2	The Finger deformation tensor	33
2.8.3	The Green-Lagrange deformation tensor	34
2.8.4	The Almansi deformation tensor	35

2.8.5	The Hencky deformation tensor	35
2.9	Representation of spatial tensors in the reference configuration (“pull-back”)	36
2.9.1	Pull-back of vector components	36
2.9.2	Pull-back of tensor components	40
2.10	Tensors in the spatial configuration from representations in the reference configuration (“push-forward”)	42
2.11	Pull-back/push-forward relations between strain measures	43
2.12	Objectivity	44
2.12.1	Reference frame and isometric transformations	45
2.12.2	Objectivity or material-frame indifference	47
2.12.3	Covariance	49
2.13	Strain rates	50
2.13.1	The velocity gradient tensor	50
2.13.2	The Eulerian strain rate tensor and the spin (vorticity) tensor	51
2.13.3	Relations between different rate tensors	53
2.14	The Lie derivative	56
2.14.1	Objective rates and Lie derivatives	58
2.15	Compatibility	61
3	Stress Tensor	67
3.1	External forces	67
3.2	The Cauchy stress tensor	69
3.2.1	Symmetry of the Cauchy stress tensor (Cauchy Theorem)	71
3.3	Conjugate stress/strain rate measures	72
3.3.1	The Kirchhoff stress tensor	74
3.3.2	The first Piola-Kirchhoff stress tensor	74
3.3.3	The second Piola-Kirchhoff stress tensor	76
3.3.4	A stress tensor energy conjugate to the time derivative of the Hencky strain tensor	79
3.4	Objective stress rates	81
4	Balance principles	85
4.1	Reynolds’ transport theorem	85
4.1.1	Generalized Reynolds’ transport theorem	88
4.1.2	The transport theorem and discontinuity surfaces	90
4.2	Mass-conservation principle	93
4.2.1	Eulerian (spatial) formulation of the mass-conservation principle	93
4.2.2	Lagrangian (material) formulation of the mass conservation principle	95
4.3	Balance of momentum principle (Equilibrium)	95

4.3.1	Eulerian (spatial) formulation of the balance of momentum principle	96
4.3.2	Lagrangian (material) formulation of the balance of momentum principle	103
4.4	Balance of moment of momentum principle (Equilibrium)	105
4.4.1	Eulerian (spatial) formulation of the balance of moment of momentum principle	105
4.4.2	Symmetry of Eulerian and Lagrangian stress measures	107
4.5	Energy balance (First Law of Thermodynamics)	109
4.5.1	Eulerian (spatial) formulation of the energy balance	109
4.5.2	Lagrangian (material) formulation of the energy balance	112
5	Constitutive relations	115
5.1	Fundamentals for formulating constitutive relations	116
5.1.1	Principle of equipresence	116
5.1.2	Principle of material-frame indifference	116
5.1.3	Application to the case of a continuum theory restricted to mechanical variables	116
5.2	Constitutive relations in solid mechanics: purely mechanical formulations	120
5.2.1	Hyperelastic material models	121
5.2.2	A simple hyperelastic material model	122
5.2.3	Other simple hyperelastic material models	128
5.2.4	Ogden hyperelastic material models	129
5.2.5	Elastoplastic material model under infinitesimal strains	135
5.2.6	Elastoplastic material model under finite strains	155
5.3	Constitutive relations in solid mechanics: thermoelastoplastic formulations	167
5.3.1	The isotropic thermoelastic constitutive model	167
5.3.2	A thermoelastoplastic constitutive model	170
5.4	Viscoplasticity	176
5.5	Newtonian fluids	180
5.5.1	The no-slip condition	181
6	Variational methods	183
6.1	The Principle of Virtual Work	183
6.2	The Principle of Virtual Work in geometrically nonlinear problems	186
6.2.1	Incremental Formulations	189
6.3	The Principle of Virtual Power	194
6.4	The Principle of Stationary Potential Energy	195
6.5	Kinematic constraints	207
6.6	Veubeke-Hu-Washizu variational principles	209
6.6.1	Kinematic constraints via the V-H-W principles	209
6.6.2	Constitutive constraints via the V-H-W principles	211

A Introduction to tensor analysis	213
A.1 Coordinates transformation	213
A.1.1 Contravariant transformation rule	214
A.1.2 Covariant transformation rule	215
A.2 Vectors	215
A.2.1 Base vectors	216
A.2.2 Covariant base vectors	216
A.2.3 Contravariant base vectors	218
A.3 Metric of a coordinates system	219
A.3.1 Cartesian coordinates	219
A.3.2 Curvilinear coordinates. Covariant metric components	220
A.3.3 Curvilinear coordinates. Contravariant metric components	220
A.3.4 Curvilinear coordinates. Mixed metric components	221
A.4 Tensors	222
A.4.1 Second-order tensors	223
A.4.2 n-order tensors	227
A.4.3 The metric tensor	228
A.4.4 The Levi-Civita tensor	229
A.5 The quotient rule	232
A.6 Covariant derivatives	233
A.6.1 Covariant derivatives of a vector	233
A.6.2 Covariant derivatives of a general tensor	236
A.7 Gradient of a tensor	237
A.8 Divergence of a tensor	238
A.9 Laplacian of a tensor	239
A.10 Rotor of a tensor	240
A.11 The Riemann-Christoffel tensor	240
A.12 The Bianchi identity	243
A.13 Physical components	244
References	247
Index	255