PREFACE XV A BRIEF GLOSSARY OF NOTATIONS XXII

Part One Linear Static Analysis

FUNDAMENTALCONCEPTS; ASIMPLEONE-DIMENSIONAL 1 **BOUNDARY-VALUE PROBLEM 1**

- 1.1 Introductory Remarks and Preliminaries
- 1.2 Strong, or Classical, Form of the Problem 2
- Weak, or Variational, Form of the Problem 1.3 3
- Eqivalence of Strong, and Weak Forms; Natural Boundary 1.4 Conditions 4 7

1

9

40

- Galerkin's Approximation Method 1.5
- Matrix Equations; Stiffness Matrix K 1.6
- 1.7 Examples: 1 and 2 Degrees of Freedom 13
- 1.8 Piecewise Linear Finite Element Space 20
- 1.9 Properties of K 22
- 1.10 Mathematical Analysis 24
- 1.11 Interlude: Gauss Elimination; Hand-calculation Version 31 37
- 1.12 The Element Point of View
- 1.13 Element Stiffness Matrix and Force Vector
- 1.14 Assembly of Global Stiffness Matrix and Force Vector; LM Array 4 2

iii

IV		Contents	Contents	
1.15	Explicit Computation of Element Stiffness Matrix and ForceVector 44		3.8 N 3.9 I	Numerical In Derivatives 0
1.16	Exercise: Bernoulli-Euler Beam Theoryand Hermitel		3.10 F	Subroutines
Appendix 1.1	An Elementary Discussion of Continuity, Differentiability,		3.11 A	Additional Ex
	References 55		Appendix 3.II N	Methodology with Applicat
2 FORMUL BOUNDA	ATIONOFTWO-ANDTHREE-DIMENSIONAL RY-VALUE PROBLEMS 57		Į	References
2.1	Introductory Remarks 57		4 MIXED AND INTEGRAT	PENALTY
2.2 2.3	Classical Linear Heat Conduction: Strong and Weak		4.1 "	Best Approxi
2.4	Heat Conduction: Galerkin Formulation; Symmetry and Positive-definiteness of K 64		d 4.2 Incon	loes not 1 npressible
2.5	Heat Conduction: Element Stiffness Matrix and Force Vector 69		4.2.1 4.3 A	Prelude ∎ Mixed Form
2.6	Heat Conduction: Data Processing Arrays ID, IEN , and LM 71		0 4.3.1	f Representin StrongFc
2.7	Classical Linear Elastostatics: Strong and Weak Forms; Equivalence 75		4.3.2 4.3	Weak For . 3 Galerkin
2.8	Elastostatics: Galerkin Formulation, Symmetry, and Positive-definiteness of K 84		4.3.4 4.3.5	Matrix P Definition
2.9	Elastostatics: Element Stiffness Matrix and Force Vector 90		4 . 3 4.3.7	. 6 Illustratio Constrain
2.10	Elastostatics: Data Processing Arrays ID, IEN , and LM 92		4.3 4.3.9	Continuou
2.11	Summary of Important Equations for Problems Considered in Chapters 1 and 2 98		4.4 Pena T	echniques; Eq
2.12	Axisymmetric Formulations and Additional Exercises 101		4.4.1 4.5 A	n Extension of
	References 107		4.5	. 1 Axisymme Analysis
3 ISOPARA PROGR	METRIC ELEMENTS AND ELEMENTARY AMMING CONCEPTS 108		4.5.2 4.6 T	Strain Proj he Patch Test:
3.1 3.2	Preliminary Concepts 109 Bilinear Ouadrilateral Element 112		487 Hourglass	Nonconforming Stiffn
3.3 3.4	Isoparametric Elements 118 Linear Triangular Element; An Example		4.9 Average	dditional Exercitational Exercitation Exerci
3.5	of "Degeneration*' 120 Trilinear Hexahedral Element 123 126		4.1.1 4.1.2	Basic Prop Sobolev No
3.6 3.7	Higher-order Elements; Lagrange Polynomials 132 Elements with Variable Numbers of Nodes		4.1.3	Approxima in Sobolev

5 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	.8	Numerical Integration; Gaussian Quadrature 137 Derivatives of Shape Functions and Shape Function Subroutines 146
	R 10	Flement Stiffness Formulation 151
9	R.11	Additional Exercises 156
Annendix	31	Triangular and Tetrahedral Elements 164
Annondiv	3 11	Mathodology for Davaloning Spacial Shane Functions
Арренних	5.11	with Application to Singularities 175
		Deferences 100
		References 182
4 M IN	IXED A TEGR	AND PENALTY METHODS, STATIONAL CRIMES 185
4	l.1	"Best Approximation" and Error Estimates: Why the stan-
		dard FEM usually works and why sometimes it
		does not 185
4	1.2	Incompressible Elasticity and Stokes Flow 192
2	1.3 4.2	2.1 Prelude to Mixed and Penalty Methods 194
		A Mixed Formulation of Compressible Elasticity Capable
		of Representing the Incompressible Limit 197
	4.3	L1 Strong Fonn 198
	4.:	3.2 Weak Form 198
	4.3	3.3 Galerkin Formulation 200
	4.3	3.4 Matrix Problem 200
	4.,	3.6 Definition of a Fundamental Difficulty
	4.5	Constraint Counts 200
	4.3	2.9 Discontinuous Pressure Flements 910
	4.3	3 9 Continuous Pressure Elements 210
4	4	Penalty Formulation: Reduced and Selective Integration
		Techniques: Equivalence with Mixed Methods 217
	4.4	1.1 Pressure Smoothing 226
4	4.5	An Extension of Reduced and Selective Integration
		Techniques 232
	4.5	5.1 Axisymmetry and Anisotropy: Prelude to Nonlinear
		Analysis 232
	4.5	5.2 Strain Projection: The B-approach 232
4	4.6	The Patch Test; Rank Deficiency 237
4	4.7	Nonconforming Elements 242
4	4.8	Hourglass Stiffness 251
4	4.9	Additional Exercises and Projects 254
Appendix	4.1	Mathematical Preliminaries
	4.1	1.1 Basic Properties of Linear Spaces 263
	4.1	1.2 Sobolev Norms 266
	4.1	1.3 Approximation Properties of Finite Element Spaces in Sobolev Norms 268

4.1.4Hypotheses on $a(\cdot, \cdot)$ 273Appendix 4.IIAdvanced Topics in the Theory of Mixed and Penalty Methods: Pressure Modes and Error Estimates2764.11.1Pressure Modes, Suritous and Otherwisel2764.11.2Existence and Uniqueness of Solutions in the Pres- ence of Modes2784.11.4Pressure Modes and Error Estimates2764.11.4Pressure Modes, 2785.4.64.11.4Pressure Modes and Dispersive Modes2814.11.5Error Estimates and Pressure Smoothing2974.11.5Error Estimates and Pressure Smoothing2978.1Introduction 3102.25.2Reitsner-Mindlin Plate Theory 3105.2.1Mait Assumptions 3105.2.2Strain-displacement Equations 3135.2.3Strain-displacement Equations 3145.2.9Frainte Element 3175.2.2Strain Element 3175.2.3Form Ja175.2.4Retiffress Mairx and Load Vector 3205.3Poundary Conditions 3245.3.4Reduced and Selective Integration Lagrange Plate Element 3275.3.5Equivalence with Mixed Methods 3305.3.6Sand Ti: A Correct-rank, Four-node Bilinear Element 3425.3.10The Discrete Kirchhoff Approach 3595.3.11Discussion of Some Quadrilateral Bending Elements 3625.4Beamas 363	vi		Contents	Contents	
Appendix 4.IIAdvanced Topics in the Theory of Mixed and Penalty Methods: Pressure Modes and Error Estimates276 2764.11.1Pressure Modes, Spurious and Otherwisel2764.11.2Pressure Modes, Spurious and Otherwisel2764.11.3Pressure Modes2784.11.4Pressure Modes2814.11.5The Big Picture2924.11.6Error Estimates and Pressure Smoothing2974.11.6Error Estimates and Pressure Smoothing2977References3035THE CO-APPROACH TO PLATES AND BEAMS3105.1Introduction 3103135.2Reissner-Mindlin Plate Theory 3105.2.1Mairis Pormulation 3135.2.3Strain-displacement Equations 3145.2.5Variational Equations 3145.2.7Weak Form 3175.2.8Kine Element Siffners Matrix and LoadVector3205.3Plate-bending Elements 3275.3.1Some Convergence Criteria 3225.3.3Fiber Num Mixed Methods 3305.3.4Reduced and Selective Integration Lagrange Plate5.3.5Equivalence with Mixed Methods 3305.3.6Gam At 25.3.10The Hetrosis Element 3355.3.10The Metrohoff Approach 3595.3.10The Discrete Kirchhoff Approach 3595.3.10The Linear Triangle 3555.3.10Gometric 6335.4Beaman and Frames 363	4.1.	4 Hypotheses on $a(\cdot, \cdot)$ 273		5.4.4	Definitio
Methods: Pressure Modes and Error Estimates 276 54.5 Variationby David S. Markus 5.45 Variationby David S. Markus 5.45 Variationby David S. Markus 276 $5.4.6$ 4.11.2Existence and Uniqueness of Solutions in the Pres- $5.4.5$ ence of Modes 278 $5.4.5$ 4.11.3Two Sides of Pressure Modes 281 4.11.4Pressure Modes in the Penalty Formulation 289 4.11.5The Big Picture 292 4.11.6Error Estimates and Pressure Smoothing 297 References 303 $5.4.5$ 5.1Introduction 310 5.2Reissner-Mindlin Plate Theory 310 5.2.1Main Assumptions 313 5.2.2Constitutive Equation 313 5.2.3Strain-displacement Equations 314 5.2.4Sumary of Plate Theory Notations 314 5.2.5Variation 319 5.2.6Strong Form5.2.7Weak Form 317 5.2.8Matrix Formulation 319 5.2.9Finite Element 316 5.3.1Soundary Conditions 324 5.3.3Boundary Conditions 324 5.3.4Reduced and Selective Integration Lagrange PlateElements 327 5.3.5Gank Deficiency 332 5.3.6Rank Deficiency 332 5.3.7The Actorect-rank, Four-node BilinearElements 327 5.3.10The Discrete Kirchhoff Approach 359 5.3.11Discussion of Some Quadrilateral BendingElemen	Appendix 4 II	Advanced Topics in the Theory of Mixed and Penalty			in the Th
by David S. Malkas 4.11.1 Pressure Modes, Spurious and Otherwisel 276 4.11.2 Existence and Uniqueness of Solutions in the Pres- ence of Modes 278 4.11.3 Two Sides of Pressure Modes in the Penalty Formulation 289 4.11.5 The Big Picture 292 4.11.5 The Big Picture 292 4.11.5 The Big Picture 292 5.1 Introduction 310 5.2 Reissner-Mindlin Plate Theory 310 5.2.1 Main Assumptions 310 5.2.2 Constitutive Equations 313 5.2.5 Variational Equations 313 5.2.6 Strong Form 317 5.2.9 Finite Element Suffress Matrix and Load Vector 320 5.3 Plate-bending Elements 327 5.3.1 Some Convergence Criteria 322 5.3.6 Rank Deficiency 323 5.3.7 The Heterosis Element 335 5.3.8 Fair: A Correct-rank, Four-node Bilinear Elements 362 5.3 The Linear Triangle 355 5.3.1 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frances 363 5.5 Reduced and Steven Bilinear Elements 362 5.4 Beams and Frances 363 5.5 Reduced and Steven Bilinear Elements 362 5.4 Beams and Frances 363 5.5 Reduced and Steven Bilinear Elements 362 5.4 Beams and Frances 363 5.5 Reduced and Steven Bilinear Elements 362 5.4 Beams and Frances 363 5.5 Reduced and Steven Bilinear Elements 362 5.5 Reduced Assective Integration Lagrange Plate Elements 362 5.5 Reduced Assective In		Methods: Pressure Modes and Error Estimates 276		5.4.5	Variation
 4.11.1 Pressure Modes, Spurious and Otherwisel 276 4.11.2 Existence and Uniqueness of Solutions in the Preserve Modes 278 4.11.3 Two Sides of Pressure Modes 278 4.11.4 Pressure Modes 278 4.11.3 Two Sides of Pressure Modes 281 4.11.4 Pressure Modes 282 4.11.5 The Big Picture 292 4.11.6 Error Estimates and Pressure Smoothing 297 5 THE CO-APPROACH TO PLATES AND BEAMS 310 5.1 Introduction 310 5.2 Retissner-Mindlin Plate Theory 310 5.2.1 Main Assumptions 310 5.2.2 Constitutive Equation 313 5.2.3 Strain-displacement Equations 314 5.2.9 Finite Element Suffness Matrix and Load Vector 320 5.3.1 Some Convergence Criteria 322 5.3.1 Some Convergence Criteria 322 5.3.4 Soundary Conditions 324 5.3.5 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 Fuel-meat mining 355 5.3.9 The Linear Triangle 355 5.3.10 The Discrets Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 		by David S. Malkus		5.4.6	Strong E
 4.11.2 Existence and Uniqueness of Solutions in the Presence of Modes 278 4.11.3 Two Sides of Pressure Modes 181 4.11.4 Pressure Modes in the Penalty Formulation 289 4.11.5 The Big Picture 292 4.11.6 Error Estimates and Pressure Smoothing 297 References 303 5 THE CO-APPROACH TO PLATES AND BEAMS 310 5.1 Introduction 310 5.2 Reitsnere Equation 313 5.2.3 Strain-displacement Equations 313 5.2.4 Summary of Plate Theory Notations 314 5.2.5 Variational Equation 314 5.2.6 Strong Form 317 5.2.8 Matrix Permulation 319 5.3 Plate-bending Elements 327 5.3.1 Some Convergence Criteria 322 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.5 Reduced Integration 323 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Elements 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	4.11	1 Pressure Modes, Spurious and Otherwise 276		547	Weak For
ence of Modes 278 4.II.3 Two Sides of Pressure Modes 281 4.II.4 Pressure Modes 129 4.II.5 The Big Picture 292 4.115 The Big Picture 292 4.116 Error Estimates and Pressure Smoothing 297 References 303 5 THE CO-APPROACH TO PLATES AND BEAMS 310 5.2 Retissner-Mindlin Plate Theory 310 5.2 Retissner-Mindlin Plate Theory 310 5.2 Retissner-Mindlin Plate Theory 310 5.2.1 Main Assumptions 313 5.2.3 Strain-displacement Equations 314 5.2.5 Variational Equation 313 5.2.6 Strong Form 317 5.2.7 Weak Form 317 5.2.8 Matrix Formulation 319 5.2.9 Finite Element Sines Matrix and Load Vector 320 5.3 Plate-bending Elements 322" 5.3 Reduced and Sterik Plate Theory 320 5.3 Reture 20 5.4 Deams and Frames 363 5.5 Reture 20 5.6 Baams and Frames 363 5.6 Reture 20 5.7 Weak Form 317 5.2.8 Matrix Formulation 319 5.3.1 Discussion of Some Quadrilateral Bending Elements 362 5.3 Big Elements 363 5.3 Plate-bending Element 362 5.3 Plate-bending Element 362 5.3 Plate-bending Element 362 5.3 Plate-bending Elements 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 363 5.4 Beams and Frames 363 5.5 Equivalence with Mixed Methods 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 5.5 Equivalence with Sifter Some Some Some Some Some Some Some Some	4.11	2 Existence and Uniqueness of Solutions in the Pres-		548	Matrix E
 4.II.3 Two Sides of Pressure Modes 281 4.II.4 Pressure Modes in the Penalty Formulation 289 4.II.5 The Big Picture 292 4.116 Error Estimates and Pressure Smoothing 297 References 303 5 THE CO-APPROACH TO PLATES AND BEAMS 310 5.1 Introduction 310 5.2 Reissner-Mindlin Plate Theory 310 5.2.1 Main Assumptions 310 5.2 Constitutive Equation 313 5.2 Strain-displacement Equations 313 5.2.3 Strain-displacement Stiffness Matrix and Load Vector 320 5.3 Plate-bending Elements 327 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 Th: A Correct-rank, Forn-node Billear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrets Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 		ence of Modes 278		0.4.0	Equation
4.II.4Pressure Modes in the Penalty Formulation289Vector4.11.5The Big Picture2924.11.6Error Estimates and Pressure Smoothing297References303207References3035.55THE CO-APPROACH TO PLATES AND BEAMS3105.1Introduction 3105.25.2Reissner-Mindlin Plate Theory 3105.2.1 Main Assumptions 3105.2.2Constitutive Equation 3135.2.3 Strain-displacement Equations 3145.2.3Strain displacement Equations 3146.25.2.5Variational Equation 3146.2.15.2.6Strong Form 3176.2.35.2.7Weak Form 3175.2.8Finite Element Stiffness Matrix and LoadVector 3206.2.65.3Plate-bending Elements5.3.1Some Convergence Criteria 3225.3.3Boundary Conditions 3245.3.4Reduced and Selective Integration Lagrange PlateElement 3276.3.15.3.9The Actorect-rank, Four-node Bilinear Element 3425.3.9The Linear Triangle 3555.3.10The Discrete Kirchhoff Approach 3595.3.11Discussion of Some Quadrilateral Bending Elements 3625.4Beams and Frames 363	4.II	3 Two Sides of Pressure Modes 281		549	Finite Ele
4.11.5The Big Picture2925.4.10Represent Coordina4.11.6Error Estimates and Pressure Smoothing2975.4.10Represent Coordina5.1Introduction 3103105.2Reissner-Mindlin Plate Theory 3105.2.1 Main Assumptions 3105.2.3 Strain-displacement Equations 3135.2.3 Strain-displacement Equations 3145.2.3 Strain-displacement Equations 3146.2Doubly Curved5.2.6Strain - displacement Equation 3176.2.1Introduction6.2Doubly Curved5.2.7Weak Form 3176.2.2Lamina C6.2.6Strain-displacement Equation 3145.2.9Finite Element Stiffness Matrix and Load Vector 3205.3Filer Element Stiffness Matrix and Load Vector 3206.2.7Strain-displacement 322"''5.3.1Some Convergence Criteria 3226.2.8External F5.3.2Shear Constraints and Locking 3236.2.9Fiber Nun5.3.3Soundary Conditions 3246.2.10Stress Res5.3.4Reduced and Selective Integration Lagrange Plate Element 3426.3.10Stress Res5.3.10The Discrets Kirchhoff Approach 3596.3.1Geometric5.3.11Discussion of Some Quadrilateral Bending Elements 3626.3.6Stress Res5.4Beams and Frames 3636.3Stress Res	4.II	4 Pressure Modes in the Penalty Formulation 289			Vector
4.11.6Error Estimates and Pressure Smoothing297CororinaReferences3035.1Cororina5THE CO-APPROACH TO PLATES AND BEAMS3105.1Introduction 3103105.2Reissner-Mindlin Plate Theory 3105.2.1 Main Assumptions 3135.2.2Constitutive Equation 3135.2.3 Strain-displacement Equations 3135.2.3Strain-displacement Equations 3146.15.2.5Variational Equation 3146.25.2.6Strong Form 3176.2.25.2.8Matrix Form 3175.2.9Finite Element Stiffness Matrix and LoadVector 3206.2.45.3Plate-bending Elements5.3.1Some Convergence Criteria 3225.3.3Boundary Conditions 3245.3.4Reduced and Selective Integration Lagrange Plate5.3.55.3.75.3.6Rank Deficiency5.3.7The Leinear Triangle5.3.871: A Correct-rank, Four-node BilinearElement 3425.3.10The Discrets Kirchhoff Approach5.3.11Discussion of Some Quadrilateral BendingElements 3625.4Beams and Frames 363	4.11	5 The Big Picture 292		5410) Represent
References3033005THE CO-APPROACH TO PLATES AND BEAMS3105.1Introduction 3105.2Reissner-Mindlin Plate Theory 3105.2.1 Main Assumptions 3105.2.1 Main Assumptions 3135.2.2 Constitutive Equations 3136.15.2.4 Summary of Plate Theory Notations 3146.25.2.5 Variational Equation 3146.25.2.6 Strong Form 3176.2.35.2.7 Weak Form 3176.2.35.2.8 Matrix Formulation 3196.2.45.2.9 Finite Element Stiffness Matrix and Load Vector 3206.2.65.3Plate-bending Elements5.3.1 Some Convergence Criteria 3226.2.85.3.4 Reduced and Selective Integration Lagrange Plate Elements 3276.35.3.5 Equivalence with Mixed Methods 3306.3.115.3.6 Rank Deficiency 3326.35.3.7 The Heterosis Element 3356.35.3.8 Ti: A Correct-rank, Four-node Bilinear Element 3426.3.15.3.9 The Linear Triangle 3556.35.3.10 The Discrete Kirchhoff Approach 3596.3.35.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.65.4Beams and Frames 3636.3.6	4.11	.6 Error Estimates and Pressure Smoothing 297		0.4.10	Coordinat
5THE CO-APPROACH TO PLATES AND BEAMS310References5.1Introduction 3105.2Reissner-Mindlin Plate Theory 310THE C°-APPROACH TO 15.2.1Main Assumptions 3105.2.2Constitutive Equations 3136.1Introduction5.2.3Strain-displacement Equations 3136.1Introduction6.2Doubly Curved5.2.4Summary of Plate Theory Notations 3146.2Doubly Curved6.2.1Geometry5.2.5Variational Equation 3196.2.4Kinematic6.2.3Fiber Coc5.2.6Storong Form 3176.2.3Fiber Coc6.2.6Strain-displacement5.2.9Finite Element Stiffness Matrix and Load Vector 3206.2.6Strain-displacement6.2.7Stiffness5.3Plate-bending Elements322"''6.2.8External H6.2.10Stress Res5.3Plate-bending Elements3226.2.9Fiber Nur5.3.4Reduced and Selective Integration Lagrange Plate Elements 3276.2.10Stress Res5.3.5Equivalence with Mixed Methods 3306.2.13Simplifica Element 3426.3.1Geometry5.3.6The Discrete Kirchhoff Approach 3596.3.1Geometry6.3.2Reduced C5.3.1Discussion of Some Quadrilateral Bending Elements 3626.3.6Stress Res6.3.6Stress Res5.4Beams and Frames 3635.3Stress Res6.3.6Stress Res		References 303	al a	5.5	Reduced Integr
 5 THE CO-APPROACH TO PLATES AND BEAMS 310 5.1 Introduction 310 5.2 Reissner-Mindlin Plate Theory 310 5.2.1 Main Assumptions 310 5.2.2 Constitutive Equation 313 5.2.3 Strain-displacement Equations 313 5.2.4 Summary of Plate Theory Notations 314 5.2.5 Variational Equation 314 5.2.6 Strong Form 317 5.2.7 Weak Form 317 5.2.8 Matrix Formulation 319 5.2.9 Finite Element Stiffness Matrix and Load Vector 320 5.3 Plate-bending Elements 322" 5.3.1 Some Convergence Criteria 322 5.3.1 Some Convergence Criteria 322 5.3.4 Reduced and Selective Integration Lagrange Plate Element 342 5.3.6 Rank Deficiency 332 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 363 					References
 5.1 Introduction 310 5.2 Reissner-Mindlin Plate Theory 310 5.2.1 Main Assumptions 310 5.2.2 Constitutive Equation 313 5.2.3 Strain-displacement Equations 313 5.2.3 Strain-displacement Equations 314 5.2.5 Variational Equation 314 5.2.6 Strong Form 317 5.2.7 Weak Form 317 5.2.8 Matrix Formulation 319 5.2.9 Finite Element Stiffness Matrix and Load Vector 320 5.3 Plate-bending Elements 322" 5.3.1 Some Convergence Criteria 322 5.3.2 Shear Constraints and Locking 323 5.3 Boundary Conditions 324 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	5 THE CO-	APPROACH TO PLATES AND BEAMS 310			teless Marride
 5.2 Reissner-Mindlin Plate Theory 310 5.2.1 Main Assumptions 310 5.2.2 Constitutive Equation 313 5.2.3 Strain-displacement Equations 313 5.2.4 Summary of Plate Theory Notations 314 5.2.5 Variational Equation 314 5.2.6 Strong Form 317 5.2.7 Weak Form 317 5.2.8 Matrix Formulation 319 5.2.9 Finite Elements Siffness Matrix and Load Vector 320 5.3 Plate-bending Elements 322" 5.3.1 Some Convergence Criteria 322 5.3.4 Reduced and Selective Integration Lagrange Plate 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	5.1	Introduction 310			References.
5.2.1 Main Assumptions 310THE C ² -APPROACH TO5.2.2 Constitutive Equations 3135.2.3 Strain-displacement Equations 3135.2.3 Strain-displacement Equations 3146.1 Introduction5.2.4 Summary of Plate Theory Notations 3146.2 Doubly Curved5.2.5 Variational Equation 3146.2.1 Geometry5.2.6 Strong Form 3176.2.2 Lamina C5.2.7 Weak Form 3176.2.3 Fiber Co5.2.8 Matrix Formulation 3196.2.4 Kinematic5.2.9 Finite Element Stiffness Matrix and Load6.2.6 Strain-disVector 3206.2.7 Stiffness5.3 Plate-bending Elements 322"'6.2.8 External F5.3.1 Some Convergence Criteria 3226.2.8 External F5.3.2 Shear Constraints and Locking 3236.2.9 Fiber Nun5.3.3 Boundary Conditions 3246.2.10 Stress Res5.3.4 Reduced and Selective Integration Lagrange Plate6.2.11 Shell Elements5.3.5 Equivalence with Mixed Methods 3306.3.1 Geometric5.3.6 Rank Deficiency 3326.35.3.7 The Heterosis Element 3356.35.3.8 T1: A Correct-rank, Four-node Bilinear Element 3426.3.1 Geometric5.3.9 The Linear Triangle 3556.3.3 Strain-disp5.3.10 The Discrete Kirchhoff Approach 3596.3.3 Strain-disp5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.6 Stress Res5.4 Beams and Frames 3636.3.6 Stress Res	5.2	Reissner-Mindlin Plate Theory 310			
5.2.2 Constitutive Équation 313ELEMENTS3835.2.3 Strain-displacement Equations3146.1Introduction5.2.4 Summary of Plate Theory Notations3146.2Doubly Curved5.2.5 Variational Equation 3146.2Lamina C5.2.6 Strong Form 3176.2.2Lamina C5.2.7 Weak Form 3176.2.3Fiber Coc5.2.8 Matrix Formulation 3196.2.4Kinematic5.2.9 Finite Element Stiffness Matrix and Load6.2.6Strain-dispvector 320322"''6.2.7Stiffness5.3 Plate-bending Elements322"'6.2.9Fiber Nur5.3.1 Some Convergence Criteria 3226.2.9Fiber Nur5.3.3 Boundary Conditions 3246.2.10Stress Res5.3.4 Reduced and Selective Integration Lagrange Plate6.2.11Shell Elements5.3.5 Equivalence with Mixed Methods 3306.2.13Simplifica5.3.6 Rank Deficiency 3326.3Shells of Revolu5.3.7 The Heterosis Element 3356.3Shells of Revolu5.3.9 The Linear Triangle 3556.3.1Geometric5.3.10 The Discrete Kirchhoff Approach 3596.3.3Strain-disp5.3.11 Discussion of Some Quadrilateral Bending6.3.4Stiffness M6.3.11 Discussion of Some Quadrilateral Bending6.3.4Stiffness M6.3.4Stiffness M6.3.5External Fo5.3.5Elements 3626.3.6Stress Res5.4Beams and Frames 3636.3.6Stress Res	5.2	2.1 Main Assumptions 310		THE Cº-AF	PROACH TO
5.2.3 Strain-displacement Equations 3136.1 Introduction5.2.4 Summary of Plate Theory Notations 3146.2 Doubly Curved5.2.5 Variational Equation 3146.2 Geometry5.2.6 Strong Form 3176.2.1 Geometry5.2.7 Weak Form 3176.2.3 Fiber Coo5.2.8 Matrix Formulation 3196.2.4 Kinematic5.2.9 Finite Element Stiffness Matrix and Load6.2.6 Strain-disVector 3206.2.6 Strain-dis5.3 Plate-bending Elements322"'5.3.1 Some Convergence Criteria 3226.2.8 External5.3.3 Boundary Conditions 3246.2.10 Stress Res5.3.4 Reduced and Selective Integration Lagrange Plate6.2.11 Shell ElementsElements 3276.2.10 Stress Res5.3.5 Equivalence with Mixed Methods 3306.2.13 Simplifica5.3.6 Rank Deficiency 3326.3 Shells of Revolu5.3.7 The Heterosis Element 3356.3 Shells of Revolu5.3.9 The Linear Triangle 3556.3.1 Geometric5.3.10 The Discrete Kirchhoff Approach 3596.3.3 Strain-disp5.3.10 The Discrete Kirchhoff Approach 3596.3.4 Stiffness M5.3.11 Discussion of Some Quadrilateral Bending6.3.4 Stiffness MElements 3626.3.6 Stress Res5.4 Beams and Frames 3636.3.6 Stress Res	5.2	2.2 Constitutive Equation 313		ELEMENT	S 383
5.2.4 Summary of Plate Theory Notations 3146.2 Doubly Curved5.2.5 Variational Equation 3146.2.1 Geometry5.2.6 Strong Form 3176.2.2 Lamina C5.2.7 Weak Form 3176.2.3 Fiber Coc5.2.8 Matrix Formulation 3196.2.4 Kinematic5.2.9 Finite Element Stiffness Matrix and Load6.2.6 Strain-disVector 3206.2.7 Stiffness5.3 Plate-bending Elements322"'5.3.1 Some Convergence Criteria 3226.2.9 Fiber Nur5.3.3 Boundary Conditions 3246.2.10 Stress Res5.3.4 Reduced and Selective Integration Lagrange Plate6.2.11 Shell ElerElements 3276.2.13 Simplifica5.3.5 Equivalence with Mixed Methods 3306.2.13 Simplifica5.3.6 Rank Deficiency 3326.3 Shells of RevoluDimensions6.3.1 Geometric5.3.9 The Linear Triangle 3556.3.2 Reduced C5.3.10 The Discrete Kirchhoff Approach 3596.3.3 Strain-displ5.3.11 Discussion of Some Quadrilateral Bending6.3.4 Stiffness M5.4 Beams and Frames 3636.3.5 External F	5.2	.3 Strain-displacement Equations 313		6.1	Introduction
5.2.5 Variational Equation 3146.2.1 Geometry5.2.6 Strong Form 3176.2.2 Lamina 05.2.7 Weak Form 3176.2.3 Fiber Coo5.2.8 Matrix Formulation 3196.2.4 Kinematic5.2.9 Finite Element Stiffness Matrix and Load Vector 3206.2.6 Strain-dis5.3 Plate-bending Elements 322"6.2.7 Stiffness 15.3.1 Some Convergence Criteria 3226.2.8 External F5.3.2 Shear Constraints and Locking 3236.2.9 Fiber Nu5.3.3 Boundary Conditions 3246.2.9 Fiber Nu5.3.4 Reduced and Selective Integration Lagrange Plate Elements 3276.2.10 Stress Res5.3.5 Equivalence with Mixed Methods 3306.2.13 Simplifica Element 3425.3.6 Rank Deficiency 3326.3 Shells of Revolu Dimensions5.3.9 The Linear Triangle 3556.3.2 Reduced C5.3.10 The Discrete Kirchhoff Approach 3596.3.3 Strain-disp5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.5 External Fo5.4 Beams and Frames 3636.3.6 Stress Res	5.2	.4 Summary of Plate Theory Notations 314		6.2	Doubly Curved
5.2.6StrongForm3176.2.2Lamina C5.2.7WeakForm3176.2.3Fiber Coc5.2.8Matrix Formulation 3196.2.4Kinematic5.2.9Finite Element Stiffness Matrix and Load Vector 3206.2.5Reduced a5.3Plate-bending Elements322"6.2.7Strain-dis5.3.1Some Convergence Criteria 3226.2.8External F5.3.2Shear Constraints and Locking 3236.2.9Fiber Nur5.3.3Boundary Conditions 3246.2.10Stress Res5.3.4Reduced and Selective Integration Lagrange Plate Elements 3276.2.11Sthell Elements5.3.5Equivalence with Mixed Methods 3306.2.13Simplifica Elements5.3.6RankDeficiency3326.3Shells of Revolu Dimesions5.3.8T1: A Correct-rank, Four-node Bilinear Element 3426.3.1Geometric5.3.9The Linear Triangle3556.3.2Reduced CO5.3.10The Discrete Kirchhoff Approach3596.3.3Strain-disp5.3.11Discussion of Some Quadrilateral Bending Elements6.3.4Stiffness M6.3.4Stiffness M6.3.5External FO6.3.5External F6.3.6Stress Res	5.2	2.5 Variational Equation 314		6.2.1	Geometry
5.2.7 Weak Form 3176.2.3 Fiber Coc5.2.8 Matrix Formulation 3196.2.4 Kinematic5.2.9 Finite Element Stiffness Matrix and Load Vector 3206.2.6 Strain-dis5.3 Plate-bending Elements 322"'6.2.7 Stiffness5.3.1 Some Convergence Criteria 322 5.3.2 Shear Constraints and Locking 323 5.3.3 Boundary Conditions 324 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 3276.2.9 Fiber Nun5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 3426.3.1 Geometric 6.3.2 Reduced C5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.6 Stress Resu5.4Beams and Frames 3636.3.6 Stress Resu	5.2	2.6 Strong Form 317		6.2.2	Lamina C
5.2.8 Matrix Formulation 3196.2.4 Kinematic5.2.9 Finite Element Stiffness Matrix and Load Vector 3206.2.5 Reduced5.3 Plate-bending Elements322"'5.3.1 Some Convergence Criteria 3226.2.8 External F5.3.2 Shear Constraints and Locking 3236.2.9 Fiber Nur5.3.3 Boundary Conditions 3246.2.10 Stress Res5.3.4 Reduced and Selective Integration Lagrange Plate Elements 3276.2.11 Shell Elements5.3.5 Equivalence with Mixed Methods 3306.2.13 Simplifica Element 3425.3.6 Rank Deficiency 3326.3 Shells of Revolu Dimensions5.3.7 The Heterosis Element 3356.3 Shells of Revolu Dimensions5.3.9 The Linear Triangle 3556.3.1 Geometric5.3.10 The Discrete Kirchhoff Approach 3596.3.3 Strain-disp5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.6 Stress Resu5.4Beams and Frames 3636.3.6 Stress Resu	5.2	2.7 Weak Form 317		6.2.3	Fiber Coo
5.2.9Finite Element Stiffness Matrix and Load Vector 3206.2.5Reduced of 6.2.65.3Plate-bending Elements322"'6.2.6Strain-dis5.3.1Some Convergence Criteria 3226.2.8External F5.3.2Shear Constraints and Locking 3236.2.9Fiber Nur5.3.3Boundary Conditions 3246.2.10Stress Res5.3.4Reduced and Selective Integration Lagrange Plate Elements 3276.2.11Shell Elem5.3.5Equivalence with Mixed Methods 3306.2.13Simplifica5.3.6RankDeficiency332Elements5.3.7The Heterosis Element 3356.3Shells of Revolu5.3.8T1: A Correct-rank, Four-node Bilinear Element 3426.3.1Geometric5.3.10The Discrete Kirchhoff Approach 3596.3.3Strain-disp5.3.11Discrete Kirchhoff Approach 3596.3.4Stiffness M6.3.5Elements 3626.3.6Stress Resu	5.2	2.8 Matrix Formulation 319		6.2.4	Kinematio
Vector3 206.2.6Strain-dis5.3Plate-bending Elements322"'6.2.7Strain-dis5.3.1Some Convergence Criteria3226.2.8External H5.3.2Shear Constraints and Locking3236.2.9Fiber Nur5.3.3Boundary Conditions3246.2.10Stress Res5.3.4Reduced and Selective Integration Lagrange Plate6.2.11Shell ElerElements3276.2.13Simplifica5.3.5Equivalence with Mixed Methods3306.2.13Simplifica5.3.6RankDeficiency3326.3Shells of Revolu5.3.7The HeterosisElement3356.3Shells of Revolu5.3.8T1: A Correct-rank, Four-node Bilinear Element3426.3.1Geometric5.3.9The Linear Triangle3556.3.2Reduced Cd5.3.10The Discrete Kirchhoff Approach3596.3.3Strain-disp5.3.11Discussion of Some Quadrilateral Bending Elements6.3.4Stiffness M6.3.4Stiffness M6.3.5External Fo6.3.4Stiffness M6.3.6Stress Resu	5.2	.9 Finite Element Stiffness Matrix and Load		6.2.5	Reduced
 5.3 Plate-bending Elements 322" 5.3.1 Some Convergence Criteria 322 5.3.2 Shear Constraints and Locking 323 5.3.3 Boundary Conditions 324 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 		Vector 320		6.2.6	Strain-dis
 5.3.1 Some Convergence Criteria 322 5.3.2 Shear Constraints and Locking 323 5.3.3 Boundary Conditions 324 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	5.3	Plate-bending Elements 322 "		6.2.7	Stiffness 1
 5.3.2 Shear Constraints and Locking 323 5.3.3 Boundary Conditions 324 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	5.3	1 Some Convergence Criteria 322		6.2.8	External H
 5.3.3 Boundary Conditions 324 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	5.3	2 Shear Constraints and Locking 323		6.2.9	Fiber Nun
 5.3.4 Reduced and Selective Integration Lagrange Plate Elements 327 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	5.3	.3 Boundary Conditions 324		6.2.10	Stress Res
Elements 3276.2.12Some Ref5.3.5 Equivalence with Mixed Methods 3306.2.13Simplifica5.3.6 Rank Deficiency 332Elements5.3.7 The Heterosis Element 3356.3Shells of Revolu5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342Dimensions5.3.9 The Linear Triangle 3556.3.1Geometric5.3.10 The Discrete Kirchhoff Approach 3596.3.3Strain-displ5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.5External Fo5.4 Beams and Frames 3636.3.6Stress Resu	5.3.	4 Reduced and Selective Integration Lagrange Plate		6.2.11	Shell Elen
 5.3.5 Equivalence with Mixed Methods 330 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 		Elements 327		6.2.12	Some Ref
 5.3.6 Rank Deficiency 332 5.3.7 The Heterosis Element 335 5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342 5.3.9 The Linear Triangle 355 5.3.10 The Discrete Kirchhoff Approach 359 5.3.11 Discussion of Some Quadrilateral Bending Elements 362 5.4 Beams and Frames 363 	5.3	5 Equivalence with Mixed Methods 330		6.2.13	Simplifica
5.3.7 The Heterosis Élement 3356.3 Shells of Revolution5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342Dimensions5.3.9 The Linear Triangle 3556.3.1 Geometric5.3.10 The Discrete Kirchhoff Approach 3596.3.3 Strain-disple5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.5 External Fo5.4 Beams and Frames 3636.3.6 Stress Result	5.3	.6 Rank Deficiency 332			Elements
5.3.8 T1: A Correct-rank, Four-node Bilinear Element 342Dimensions5.3.9 The Linear Triangle 3556.3.1 Geometric5.3.10 The Discrete Kirchhoff Approach 3596.3.2 Reduced Co5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.4 Stiffness M5.4 Beams and Frames 3636.3.6 Stress Resu	5.3	.7 The Heterosis Element 335		6.3 S	hells of Revolution
Element 3426.3.1Geometric5.3.9The Linear Triangle 3556.3.2Reduced Co5.3.10The Discrete Kirchhoff Approach 3596.3.3Strain-displ5.3.11Discussion of Some Quadrilateral Bending Elements 3626.3.4Stiffness M5.4Beams and Frames 3636.3.6Stress Result	5.3	.8 T1: A Correct-rank, Four-node Bilinear		D	imensions
5.3.9 The Linear Triangle 3556.3.2Reduced Co5.3.10 The Discrete Kirchhoff Approach 3596.3.3Strain-displ5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.4Stiffness M5.4 Beams and Frames 3636.3.6Stress Resu		Element 342		6.3.1	Geometric
5.3.10 The Discrete Kirchhoff Approach 3596.3.3Strain-displ5.3.11 Discussion of Some Quadrilateral Bending Elements 3626.3.4Stiffness M5.4Beams and Frames 3636.3.6Stress Resu	5.3	.9 The Linear Triangle 355		6.3.2	Reduced Co
5.3.11 Discussion of Some Quadrilateral Bending6.3.4Stiffness MElements3626.3.5External Fo5.4Beams and Frames3636.3.6Stress Resu	5.3	.10 The Discrete Kirchhoff Approach 359		6.3.3	Strain-displ
Elements 362 6.3.5 External Fo 5.4 Beams and Frames 363 6.3.6 Stress Resu	5.3	.11 Discussion of Some Quadrilateral Bending		6.3.4	Stiffness M
5.4 Beams and Frames 363 6.3.6 Stress Resu		Elements 362		6.3.5	External Fo
	5.4	Beams and Frames 363		6.3.6	Stress Resu

6.3.7

6.3.8

References

Boundary C

Shell Eleme

Beams and Frames 363 5.4 5.4.1 Main Assumptions 363

5.4.2 Constitutive Equation 365

5.4.3 Strain-displacement Equations 366

Definitions of Quantities Appearing
in the Theory 366
Variational Equation 368
Strong Form 371
Weak Form 372
Matrix Formulation of the Variational
Equation 373
Finite Element Stiffness Matrix and Load
Vector 374
Representation of Stiffness and Load in Global
Coordinates 376

5.5Reduced Integration Beam Elements376References379

THE CO-APPROACH TO CURVED STRUCTURAL ELEMENTS 383

6.1 Int	roduction 383
6.2 Do	ubly Curved Shells in Three Dimensions 384
6.2.1	Geometry 384
6.2.2	Lamina Coordinate Systems 385
6.2.3	Fiber Coordinate Systems 387
6.2.4	Kinematics 388
6.2.5	Reduced Constitutive Equation 389
6.2.6	Strain-displacement Matrix 392
6.2.7	Stiffness Matrix 396
6.2.8	External Force Vector 396
6.2.9	Fiber Numerical Integration 398
6.2.10	Stress Resultants 399
6.2.11	Shell Elements 399
6.2.12	Some References to the Recent Literature 403
6.2.13	Simplifications: Shells as an Assembly of Flat
	Elements 404
6.3 She	lls of Revolution; Rings and Tubes in Two
Dii	mensions 405
6.3.1	Geometric and Kinematic Descriptions 405
6.3.2	Reduced Constitutive Equations 407
6.3.3	Strain-displacement Matrix 409
6.3.4	Stiffness Matrix 412
6.3.5	External Force Vector 412
6.3.6	Stress Resultants 413
6.3.7	Boundary Conditions 414
6.3.8	Shell Elements 414
Ref	erences 415

Viii		Contents	Contents
Part Two Line 7 FORMULATI EIGENVALUE 7.1 Pa 7.2 Hy D 7.3 Ei	ear Dynamic Analysis ON OF PARABOLIC, HYPERBOLIC, AND ELLIPTIC PROBLEMS 418 rabolic Case: Heat Equation 418 yperbolic Case: Elastodynamics and Structural ynamics 423 genvalue Problems: Frequency Analysis	4	9.2 summary of T E l e m e n t s 9.3 Linear Multiste 9.3.1 LMS Me 9.3.2 LMS Me 9.3.3 Survey of in Structu 9.3.4 Some Red Dynamics
an 7.3.1 7.3.2 7.3.3 Appendix 7.1 Er Aj R	d Buckling 429 Standard Error Estimates 433 Alternative Definitions of the Mass Matrix; Lumper and. Higher-order Mass 436 Estimation of Eigenvalues 452 ror Estimates for Semidiscrete Galerkin pproximations 456 eferences 457	d	9.4 Algorithms Bas Partitions 9.4.1 Stability 9.4.2 Predictor/ 9.5 Mass Matrices I References
8 ALGORITHM 8.1 Or 6.2 'Ar 8.2.1 8.2.2 8.2.3 8.2.4 8.2.5 8.3 E di or 8.4 Elemon 8.5 Mod R	 IS FOR PARABOLIC PROBLEMS 459 ne-step Algorithms for the Semidiscrete Heat Equation eneralized Trapezoidal Method 459 malysis of the Generalized Trapezoidal Method 46 Modal Reduction to SDGF Form 462 Stability 465 Convergence 468 An Alternative Approach to Stability: The Energy Method 471 Additional Exercises 473 Iementary Finite Difference Equations for the One-Immensional Heat Equation; the von Neumann Method f Stability Analysis 479 ent-by-element (EBE) Implicit Methods 483 Analysis 487 Analysis 487 	10 : 2	SOLUTION TECHNIQUE: PROBLEMS 570 10.1 The Generalized 10.2 Static Condensal 10043IrDissGuyane Rayleig Red 10.5 Subspace Iteration 10.5.1 Spectrum S 10.5.2 Inverse Iter 10.6 The Lanczos Alg Eigenproblems by Bahram Nour-C 10.6.1 Introduction 10.6.2 Spectral Tr 10.6.3 Conditions 10.6.4 The Rayleig Derivation 1 10.6.6 Reduction to
9 ALGORITH HYPERBOI 9.1 C 0 9.1.1 9.1.2	MSFOR HYPERBOLIC AND PARABOLIC LIC PROBLEMS 490 One-step Algorithms for the Semidiscrete Equation f Motion 490 The Newmark Method 490 Analysis 492		10.6.7 Convergend 10.6.8 Loss of Ort 10.6.9 Restoring 0 References 6
9.1.2 9.1.3 9.1.4	Measures of Accuracy: Numerical Dissipation and Dispersion 504 Matched Methods 505	11	DLEARN-A LINEAR STAT ANALYSISPROGRAM by Thomas J. R. Hughes, Rol

9.1.4 9.1.5 Additional Exercises 512 by Thomas J. R. Hughes, Rol and Arthur M. Raefsky

- 9.2 Summary of Time-step Estimates for Some Simple Finite Elements 513 Linear Multistep (LMS) Methods 523 9.3 9.3.1 LMS Methods for First-order Equations 523 9.3.2 LMS Methods for Second-order Equations 526 9.3.3 Survey of Some Commonly Used Algorithms in Structural Dynamics 529 9.3.4 Some Recently Developed Algorithms for Structural Dynamics 550 9.4 Algorithms Based upon Operator Splitting and Mesh Partitions 552 9.4.1 Stability via the Energy Method 556 9.4.2 Predictor/Multicorrector Algorithms 562 9.5 Mass Matrices for Shell Elements 564 References 567 SOLUTION TECHNIQUES FOR EIGENVALUE PROBLEMS 570 10.1 The Generalized Eigenproblem 570 10.2 Static Condensation 573 10.3 Discrete Rayleigh-Ritz Reduction 574 Irons-Guyan Reduction 10.4 576 10.5 Subspace Iteration 576 10.5.1 Spectrum Slicing 578 10.5.2 Inverse Iteration 579 10.6 The Lanczos Algorithm for Solution of Large Generalized Eigenproblems **582** by Bahram Now-Omid 10.6.1 Introduction 582 10.6.2 Spectral Transformation 583 10.6.3 Conditions for Real Eigenvalues 584 10.6.4 The Rayleigh-Ritz Approximation 585 Derivation of the Lanczos Algorithm 10.6.5 586 Reduction to Tridiagonal Form 10.6.6 589
 - **10.6.7** Convergence Criterion for Eigenvalues 592
 - **10.6.8**Loss of Grthogonality**595**
 - 10.6.9 Restoring orthogonality 598 References 601

DLEARN-A LINEAR STATIC AND DYNAMIC FINITE ELEMENT ANALYSIS PROGRAM 603 by Thomas J. R. Hughes, Robert M. Ferencz, and Arthur M. Raefsky

ix

11

10



- Examples 663 1. Planar Truss 663
 - Italian Truss 003
 Static Analysis of a Plane Strain Cantilever Beam 666
 Dynamic Analysis of a Plane Strain Cantilever
 - Beam 666 4. I m p l i
 - of a Rod 668
- **11.5.4**Subroutine Index for Program Listing**670**References**675**



5

This book is based on **cours nology** and Stanford Univers method in linear static and d

Preface

method in linear static and d and physical science studen element methodology, fron mentations

Some sections of this t used as lecture notes in a nu Europe, Japan, and the Unite to more experienced analysts

SUBJECTS COVERED

The first chapter of the book simple one-dimensional mod lations of two- and three-dime fact, all problems governed potential flow, elastic membr These serve as the basis for fir illustrate the relationship **betw** problems and their "weak," approximate solution is empha "variational principles" due tc **proach** In Chapter 3 a variety

