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# Preface

This book is based on **courses** and Stanford University method in linear static and dynamic and physical science **student** element methodology, from **mentations**.

Some sections of this text used as lecture notes in a **nu** Europe, **Japan**, and the **United** to more experienced **analysts**.

## SUBJECTS COVERED

The first chapter of the book simple one-dimensional modulations of two- and three-dimensional fact, all problems governed potential flow, elastic membrane. These serve as the basis for further illustrate the relationship **between** problems and their “weak,” approximate solution is emphasized “variational principles” due to **proach**. In Chapter 3 a variety

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