CONTENTS

I. BASIC CONCEPTS 1

1. Introduction 1
2. Basic concepts 1
3. The analysis in terms of normal modes 3
4. Non-dimensional numbers 6

BIBLIOGRAPHICAL NOTES 7

II. THE THERMAL INSTABILITY OF A LAYER OF FLUID HEATED FROM BELOW 9

1. THE BÉNARD PROBLEM

5. Introduction 9
6. The nature of the physical problem 9
7. The basic hydrodynamic equations 10
(a) The equation of continuity 10
(b) The equations of motion 11
(c) The rate of viscous dissipation 13
(d) The equation of heat conduction 14
8. The Boussinesq approximation 16
9. The perturbation equations 18
(a) The boundary conditions 21
10. The analysis into normal modes 22
(a) The solutions for the horizontal components of the velocity 24
11. The principle of the exchange of stabilities 24
12. The equations governing the marginal state and the reduction to a characteristic value problem 26
13. The variational principles 27
(a) The first variational principle 27
(b) The second variational principle 31
14. The thermodynamic significance of the variational principle 32
15. Exact solutions of the characteristic value problem 34
(a) The solution for two free boundaries 35
(b) The solution for two rigid boundaries 36
(i) The even solutions 37
(ii) The odd solutions 39
(c) The solution for one rigid and one free boundary 42
(d) Summary of the results for the three cases 43
16. The cell patterns 43
(a) Rolls 44
(b) Rectangular and square cells 44

(c) Hexagonal cells 47
(d) Triangular cells 50
(e) More general cell patterns 61
17. The variational solution 63
18. Experiments on the onset of thermal instability in fluids 69
(a) Bénard's experiments 60
(b) The Schmid-Milverton principle for detecting the onset of thermal instability 61
(c) The precision experiments of Silveston 64
(d) Observations by optical methods 69
BIBLIOGRAPHICAL NOTES 71

III. THE THERMAL INSTABILITY OF A LAYER OF FLUID HEATED FROM BELOW 76
2. THE EFFECT OF ROTATION
19. Introduction 76
20. The theorems of Helmholtz and Kelvin 76
21. The equations of hydrodynamics in a rotating frame of reference 80
22. The Taylor-Proudman theorem 83
23. The propagation of waves in a rotating fluid 86
24. The problem of thermal instability in a rotating fluid: general considerations 87
26. The perturbation equations 87
(a) The analysis into normal modes 89
26. The case when instability sets in as stationary convection. A variational principle 89
(a) A variational principle 91
27. Solutions for the case when instability sets in as stationary convection 94
(a) The solution for two free boundaries 94
(b) The solution for two rigid boundaries 97
(c) The solution for one rigid and one free boundary 101
(d) The origin of the $T^{\frac{3}{2}}$-law 104
28. The motions in the horizontal planes and the cell patterns at the onset of instability as stationary convection 106
(a) Rolls, rectangles, and squares 108
(b) Hexagons 111
(c) The limiting behaviour of the streamlines for $T \to \infty$ 113
29. On the onset of convection as overstability. The solution for the case of two free boundaries 114
(a) The nature of the roots of the characteristic equation (216) 121
30. On a method of discriminating the character of the marginal state. A general variational principle 123
(a) The variational principle 124

31. The onset of convection as overstability: the solution for other boundary conditions 126
32. The case p =O 129
33. Thermodynamic significance of the variational principles 130
(a) The case when the marginal state is stationary 131
(b) The case when the marginal state is oscillatory 132
34. The case when Ω and g act in different directions 134
36. Experiments on the onset of thermal instability in rotating fluids 135
(a) Experiments with water 136
(b) Experiments with mercury 138
BIBLIOGRAPHICAL NOTES *143*

IV. THE THERMAL INSTABILITY OF A LAYER OF FLUID HEATED FROM BELOW *146*

3. THE EFFECT OF A MAGNETIC FIELD

36. Hydromagnetics *146*
37. The basic equations of hydromagnetics *146*
38. The equation of motion governing the magnetic field and some of its consequences *148*
(a) The decay of a magnetic field in the absence of fluid motions. The Joule dissipation 149
(b) The case when there are motions and the conductivity is infinite 161
(i) Conservation theorems 162
(ii) The transformation of magnetic energy into kinetic energy and conversely 163
(c) The general energy equation *164*
39. The Alfvén waves *166*
(a) The case when $\nu = \eta = 0$ *166*
(b) The effects of finite viscosity and resistivity 157
40. Some special solutions of the hydromagnetic equations. The analogue of the Taylor–Proudman theorem *167*
(a) The equipartition solution *167*
(b) Force-free fields 168
(c) The analogue of the Taylor-Proudman theorem 169
41. The problem of thermal instability in the presence of a magnetic field: general considerations 169
42. The perturbation equations *160*
(a) The boundary conditions 162
(b) The analysis into normal modes 163
(c) The solutions for the horizontal components of the velocity and the magnetic field *164*
43. The case when instability sets in as stationary convection. A variational principle *166*
(a) A variational principle 166
(b) The thermodynamic significance of the variational principle 167

44. Solutions for the case when instability sets in as stationary convection 169
(a) The solution for two free boundaries 170
(b) The solution for two rigid boundaries 172
(c) The solution for one rigid and one free boundary 176
(d) Cell patterns 176
46. The origin of the &?-law and an invariant 177
(o) An invariant 180
46. On the onset of convection as overstability 181
47. The case when H and g act in different directions 186
48. Experiments on the inhibition of thermal convection by a magnetic field 190
BIBLIOGRAPHICAL NOTES 193

V. THE THERMAL INSTABILITY OF A LAYER OF FLUID HEATED FROM BELOW 196

4. THE EFFECT OF ROTATION AND MAGNETIC FIELD
49. The like and the contrary effects of rotation and magnetic field on fluid behaviour 196
50. The propagation of hydromagnetic waves in a rotating fluid 197
51. The perturbation equations 198
52. The case when instability sets in as stationary convection 201
(a) The solution for the case of two free boundaries 202
53. The case when instability sets in as overstability 209
(a) An approximate solution applicable to liquid metals 210
(b) Numerical results for mercury 211
54. Experiments on the onset of thermal instability in the presence of rotation and magnetic field
(a) The results on the critical Rayleigh number and on the manner of the onset of instability 213
(b) Optical observations and the discontinuous variation of the cell dimensions with the strength of the magnetic field 217
BIBLIOGRAPHICAL NOTES 219

VI. THE ONSET OF THERMAL INSTABILITY IN FLUID SPHERES AND SPHERICAL SHELLS 220
55. Introduction 220
56. The perturbation equations 220
(a) The operator L^2 222
(b) The analysis into normal modes 223
(c) Boundary conditions 224
(d) The velocity field 226
57. The validity of the principle of the exchange of stabilities for the case $\beta =$ constant and $\gamma =$ constant 226
58. A variational principle for the case when β and γ are constants 229
(a) The thermodynamic significance of the variational principle 230

69. On the onset of thermal instability in a fluid sphere 231
(a) The cell patterns 234
60. On the onset of thermal instability in spherical shells 237
(i) Free surfaces at $r = 1$ and $r = \eta$ 241
(ii) A rigid surface at $r = \eta$, and a free surface at $r = 1$ 241
(iii) A free surface at $r = \eta$, and a rigid surface at $r = 1$ 242
(iv) Rigid surfaces at $r = 1$ and $r = \eta$ 243
(a) The case $b = c = 1$ 243
(b) The case $b(r) = 1$ 247
(c) The case c(r) = 1 260
61. On the effect of rotation on the onset of thermal instability in a fluid sphere. The formulation of the problem 261
(a) The representation of an axisymmetric solenoidal vector field 262
(b) The perturbation equations 264
(c) The boundary conditions 266
(d) The variational principle 267
(e) The thermodynamic significance of the variational principle 269
62. The effect of rotation on the onset of stationary convection in a fluid sphere 260
63. Some remarks on geophysical applications 266
BIBLIOGRAPHICAL NOTES 268

VII. THE STABILITY OF COUETTE FLOW 272
64. Introduction 272
66. The physical problem 272
66. Rayleigh's criterion 273
67. Analytical discussion of the stability of inviscid Couette flow 277
(a) The equations in terms of the Lagrangian displacement 278
(b) The case $m = 0$ 280
(c) The case $m \neq 0$ 281
68. The periods of oscillation of a rotating column of liquid 284
(a) The case $\Omega =$ constant 284
(i) The case $\eta = 0$ 286
(ii) The case m = 0 287
(b) The case $\Omega = A + B/r^2$ and $m = 0$ 288
(i) The solution for a narrow gap 288
(ii) The formal solution for a wide gap 290
69. On viscous Couette flow 292
70. The perturbation equations 294
(a) The stability of the flow for $\mu > \eta^2$ 296
71. The solution for the case of a *narrow* gap when the marginal state is stationary 298
(a) The solution of the characteristic value problem for the case $\sigma = 0$ 300

(b) Numerical results 303
(c) An alternative method of solution 307
(d) An approximate solution for $\mu \to 1$ 309
(e) The asymptotic behaviour for $(1-\mu) \to \infty$ 313
72. On the principle of the exchange of stabilities 316
73. The solution for a wide gap when the marginal state is stationary 318
(a) The characteristic equation 319
(b) Numerical results for the case $\eta = \frac{1}{2}$ 321
74. Experiments on the stability of viscous flow between rotating cylinders 324
(a) The determination of the critical Taylor numbers, for the case $\mu = 0$, by torque measurements 327
(i) Results of the experiments with the narrow gap 330
(ii) Results of the experiments with the wide gap ($\eta = \frac{1}{2}$) 330
(b) The dependence of the critical Taylor number on Ω_2/Ω_1. The results of visual and photographic observations 333
(i) Observations on the wave numbers of the disturbance manifested at marginal stability 336
(if) Comparison between the measured and the predicted (T_c, μ)-relations 337
BIBLIOGRAPHICAL NOTES 339

VIII. THE STABILITY OF MORE GENERAL FLOWS BETWEEN COAXIAL CYLINDERS 343
76. Introduction 343
76. The stability of viscous flow in a curved channel 343
(a) The perturbation equations 344
(b) The solution of the characteristic value problem for the case $\sigma = 0$ 346
(c) Numerical results 348
77. The stability of viscous flow between rotating cylinders when a transverse pressure gradient is present 360
(a) The perturbation equations for the case $(R_2-R_1) \ll \frac{1}{2}(R_2+R_1)$ 361
(b) The solution of the characteristic value problem for the case $\sigma = 0$ and $\mu = 0$ 362
(c) The physical interpretation of the results 363
(d) Comparison with experimental results 368
78. The stability of inviscid flow between coaxial cylinders when an axial pressure gradient is present 369
(a) The case of a pure axial flow 361
(b) The general case when rotation is also present 366
(i) A variational principle for c 368
(ii) A variational principle for λ^2 369
(iii) The criterion for stability 369

79. The stability of viscous flow between rotating coaxial cylinders when an axial pressure gradient is present 371
(a) The perturbation equations 372
(b) The reduction to the case of a narrow gap 373
(c) An approximate solution of the characteristic value problem for the case $\mu > 0$ 374
(d) Comparison with experimental results 377
BIBLIOGRAPHICAL NOTES 379

IX. THE STABILITY OF COUETTE FLOW IN HYDROMAGNETICS 382
80. The equations of hydromagnetics in cylindrical polar coordinates 382
81. The stability of non-dissipative Couette flow when a magnetic field parallel to the axis is present 384
(a) The case m= 0 386
(b) The case $m \neq 0$ *389*
82. The periods of oscillation of a rotating column of liquid when a magnetic field is impressed in the direction of the axis 390
(a) The case Ω = constant 390
(b) The case $m = 0$ and $\Omega = A + B/r^2$; the stabilizing effect of a magnetic field in case of narrow gaps 391
83. The stability of non-dissipative Couette flow when a current flows parallel to the axis 393
(a)The case m=O 396
(b) The case $m \neq 0$ *396*
84. The stability of non-dissipative Couette flow when an axial and a transverse magnetic field are present 396
(a) The case $\Omega = 0$ 397
85. The stability of dissipative Couette flow in hydromagnetics. The perturbation equations 398
(a) The boundary conditions 400
(b) The equations governing the marginal state when the onset of instability is as a stationary secondary flow 401
(c) The reduction to the case of a narrow gap 402
86. The solution of the characteristic *value* problem for the case $\mu > 0$ 403
(a) A variational principle 404
(b) The solution of the characteristic value problem. The secular determinant 406
(c) The case of non-conducting walls 408
(d) The case of conducting walls 410
(e) Numerical results 411
(f) The asymptotic behaviour for $Q \to \infty$ 413
87. The solution of the characteristic value problem in the genera1 case 416
(a) The case $\mu = -1$ 417

88. The stability of dissipative flow in a curved channel in the presence of an axial magnetic field 422
(a) The equations governing the marginal state when the onset of instability is as a stationary secondary flow 422
(b) The stability of a pure pressure maintained flow 422
89. Experiments on the stability of viscous flow between rotating cylinders when an axial magnetic field is present 425
BIBLIOGRAPHICAL NOTES 426

X. THE STABILITY OF SUPERPOSED FLUIDS: THE RAYLEIGH–TAYLOR INSTABILITY 428
90. Introduction 428
91. The character of the equilibrium of a stratified heterogeneous fluid. The perturbation equations 428
(a) Allowance for surface tension at interfaces between fluids 430
92. The inviscid case 433
(a) The case of two uniform fluids of constant density separated by a horizontal boundary 434
(b) The case of exponentially varying density 435
93. A general variational principle 436
(a) The variational principle 440
94. The case of two uniform viscous fluids separated by a horizontal boundary 441
(a) The case $\nu_1 = \nu_2$ 443
(b) The modes of maximum instability for the case $\nu_1 = \nu_2$, $\rho_2 > \rho_1$, and $S = 0$ 444
(c) The effect of surface tension on the unstable modes for $\nu_1 = \nu_2$ and $\rho_2 > \rho_1$ 447
(d) The manner of decay in the case $\nu_1 = \nu_2$, $\rho_2 < \rho_1$, and $S = 0$ 448
(e) Gravity waves 451
95. The effect of rotation 453
(a) The case of two uniform fluids separated by a horizontal boundary 455
(b) The case of exponentially varying density 456
96. The effect of a vertical magnetic field 457
(a) Two uniform fluids separated by a horizontal boundary: the unstable case 459
(b) Two uniform fluids separated by a horizontal boundary: the stable case 463
97. The effect of a horizontal magnetic field 464
98. The oscillations of a viscous liquid globe 466
(a) The perturbation equations and their solution 467
(i) The solution for the inviscid case: the Kelvin modes 468
(ii) The solution for the general case 469
(b) The boundary conditions and the characteristic equation 470
(c) The manner of decay of the Kelvin modes. Numerical results 472

99. The oscillations of a viscous liquid drop 475
(a) The solution for the inviscid case 475
(b) The solution for the general case 476
BIBLIOGRAPHICAL NOTES 477

XI. THE STABILITY OF SUPERPOSED FLUIDS: THE KELVIN-HELMHOLTZ INSTABILITY 481
100. The perturbation equations 481
101. The case of two uniform fluids in relative horizontal motion separated by a horizontal boundary 483
(a) The case when surface tension is absent 484
(b) The stabilizing effect of surface tension 485
102. The effect of a continuous variation of U on the development of the Kelvin-Helmholtz instability 487
103. The Kelvin-Helmholta instability in a fluid in which both ρ and U are continuously variable 491
(a) Analytical results for the case $\rho = \rho_0 e^{-\beta z}$ and $U = U_0 z/d$ 492
104. An example of the instability of a shear layer in an unbounded heterogeneous inviscid fluid 494
105. The effect of rotation 498
(a) The case of two uniform fluids in relative horizontal motion separated by a horizontal boundary 499
(b) The discussion of the characteristic equation 501
106. The effect of a horizontal magnetic field 607
(a) The effect of a magnetic field in the direction of streaming 608
(i) The case of two uniform fluids in relative horizontal motion separated by a horizontal boundary 510
(b) The effect of a magnetic field transverse to the direction of streaming 511
BIBLIOGRAPHICAL NOTES 612

XII. THE STABILITY OF JETS AND CYLINDERS 615
107. Introduction 515
108. The gravitational instability of an infinite cylinder 616
(a) The origin of the gravitational instability and an alternative method of determining the characteristic frequencies 520
109. The effect of viscosity on the gravitational instability of an infinite cylinder 523
(a) The case when the effects of viscosity are dominant 527
(b) The general case 529
110. The effect of a uniform axial magnetic field on the gravitational instability of an infinite cylinder 531
(a) The nature of the stabilizing effect of the axial magnetic field 534
111. The capillary instability of a liquid jet 637
(a) The origin of the capillary instability 539

(b) The capillary instability of a hollow jet 539
(c) The effect of viscosity 540
112. The effect of a uniform axial magnetic field on the capillary instability of a liquid jet 542
(a) The effect of finite electrical conductivity 545
(b) The cas of high resistivity 549
(c) The general case 551
113. The stability of the simplest solution of the equations of hydromagnetics 551
(a) An example 554
114. The effect of fluid motions on the stability of helical magnetic fields 556
(a) The case f=0 560
(b) The case f= 1 562
(c) The general case 563
115. The stability of the pinch 565
(a) On stable pinch configurations 569
BIBLIOGAPHiCAL NOTES 574

XIII. GRAVITATIONAL EQUILIBRIUM AND GRAVITATIONAL INSTABILITY 577
116. Introduction 577
117. The virial theorem 577
(a) Some definitions and relations 577
(b) The general form of the virial theorem 579
(c) The virial theorem for equilibrium configurations 581
118. The virial theorem for small oscillations about equilibrium 583
(a) A characteristic equation for determining the periods of oscillation 586
119. The gravitational instability of an infinite homogeneous medium; Jeans's criterion 588
120. The effect of uniform rotation and a uniform magnetic field on Jeans's criterion 589
(a) The effect of a uniform rotation 589
(b) The effect of a uniform magnetic field 591
(c) The effect of the simultaneous presence of rotation and magnetic field 594
BIBLIOGRAPHICAL NOTES 596

XIV. A GENERAL VARIATIONAL PRINCIPLE 599
121. Introduction 599
122. A variational principle for treating the stability of hydromagnetic systems 599
(a) The effect of viscosity 603
123. Extension of the variational principle to allow for compressibility 605
BIBLIOGRAPHICAL NOTES 608

APPENDIX I. INTEGRAL RELATIONS GOVERNING STEADY CONVECTION 609
124. Introduction 609
125. The integral relations 609
(a) The integral relations 611
126. The amplitude of the steady convection past marginal stability 612
BIBLIOGRAPHICAL NOTES 615

APPENDIX II. THE VARIATIONAL FORMULATION OF THE PROBLEM CONSIDERED IN CHAPTER V 617
127. The general equation governing the energy-balance 617
BIBLIOGRAPHICAL NOTE 621

APPENDIX III. TOROIDAL AND POLOIDAL VECTOR FIELDS 622
128. A general characterization of solenoidal vector fields. The fundamental basis 622
129. The orthogonality properties of the basic toroidal and poloidal fields 623
BIBLIOGRAPHICAL NOTES 626

APPENDIX IV. VARIATIONAL METHODS BASED ON ADJOINT DIFFERENTIAL SYSTEMS 627
130. Adjointness of differential systems. An example 627
131. The dual relationship and the variational principle 629
BIBLIOGRAPHICAL NOTES 633

APPENDIX V. ORTHOGONAL FUNCTIONS WHICH SATISFY FOUR BOUNDARY CONDITIONS 634
132. Introduction 634
133. Functions suitable for problems with plane boundaries 634
(a) Integrals involving $C_n(x)$ and $S_m(x)$ 636
134. Functions suitable for problems with cylindrical and spherical boundaries 637
(a) The normalization integral 638
(b) The cylinder functions of order $\frac{1}{2}$ 640
(c) The spherical functions of half-odd integral orders 640
BIBLIOGRAPHICAL NOTES 642

SUBJECT INDEX 645

INDEX OF DEFINITIONS 653