

Contents

List of Tablesx v
Prefacexv i

Part I. Probability Spaces, Random Variables, and Expectations

Chapter 1. Probability Spaces	3
1.1. Introductory examples	·
1.2. Ingredients of probability spaces	6
1.3. σ -fields	8
1.4. Borel-fields	·
Chapter 2. Random Variables	11
2.1. Definitions and basic results	11
2.2. \mathbb{R}^d -valued random variables	14
2.3. \mathbb{R}^∞ -valued random variables	18
2.4. Further examples	20
Chapter 3. Distribution Functions	25
3.1. Basic theory	25
3.2. Examples of distributions	29
3.3. Some descriptive terminology	31
3.4. Distributions with densities	35
3.5. Further examples	37
3.6. Distribution functions for the extended real line	39

Chapter 4. Expectations: Theory	41
4.1. Definitions	41
4.2. Linearity and positivity	46
4.3. Monotone convergence	49
4.4. Expectation of compositions	51
4.5. The Riemann-Stieltjes integral and expectations	53
Chapter 5. Expectations: Applications	59
5.1. Variance and the Law of Large Numbers	59
5.2. Mean vectors and covariance matrices	66
5.3. Moments and the Jensen Inequality	68
5.4. Probability generating functions	70
5.5. Characterization of probability generating functions	73
Chapter 6. Calculating Probabilities and Measures	75
6.1. Operations on events	75
6.2. The Borel-Cantelli and Kochen-Stone Lemmas	77
6.3. Inclusion-exclusion	80
6.4. Finite and σ -finite measures	81
Chapter 7. Measure Theory: Existence and Uniqueness	85
7.1. The Sierpinski Class Theorem and uniqueness	85
7.2. Finitely additive functions defined on fields	87
7.3. Existence, extension, and completion of measures	90
7.4. Examples	95
Chapter 8. Integration Theory	101
8.1. Lebesgue integration	101
8.2. Convergence theorems	105
8.3. Probability measures and infinite measures compared	110
8.4. Lebesgue integrals and Riemann-Stieltjes integrals	111
8.5. Absolute continuity and densities	115
8.6. Integration with respect to counting measure	118

Part 2. Independence and Sums

Chapter 9. Stochastic Independence	121
9.1. Definition and basic properties	121
9.2. Product measure: finitely many factors	127
9.3. The Fubini Theorem	130
9.4. Expectations and independence	133
9.5. Densities and independence	134
9.6. Product probability measure: infinitely many factors	136
9.7. The Borel-Cantelli Lemma and independent sequences	141
9.8. † Order statistics	143
9.9. † Some new distributions involving independence	145
Chapter 10. Sums of Independent Random Variables	147
10.1. Convolutions of distributions	147
10.2. Multinomial distributions	152
10.3. Probability generating functions and sums in $\overline{\mathbb{Z}}^+$	153
10.4. ‡ Dirichlet distributions	156
10.5. † Random sums in various settings	159
Chapter 11. Random Walk	163
11.1. Random sequences	163
11.2. Definition and examples	164
11.3. Filtrations and stopping times	171
11.4. Stopping times and random walks	174
11.5. A hitting-time example	175
11.6. Returns to 0	178
11.7. † Random walks in various settings	183
Chapter 12. Theorems of A.S. Convergence	185
12.1. Convergence in probability	185
12.2. Laws of Large Numbers	188
12.3. Applications	193
12.4. 0-1 laws	196
12.5. Random infinite series	198

12.6. The Etemadi Lemma	200
12.7. † The Kolmogorov Three-Series Theorem	202
12.8. † The image of a random walk	206
Chapter 13. Characteristic Functions	209
13.1. Definition and basic examples	209
13.2. The Parseval Relation and uniqueness	211
13.3. Characteristic functions of convolutions	214
13.4. Symmetrization	217
13.5. Moment generating functions	218
13.6. Moment theorems	223
13.7. Inversion theorems	226
13.8. Characteristic functions in \mathbb{R}^d	234
13.9. Normal distributions on d-dimensional space	237
13.10. † An application to random walks on \mathbb{Z}	238
13.11. † An application to the calculation of a sum	239
Part 3. Convergence in Distribution	
Chapter 14. Convergence in Distribution on the Real Line	243
14.1. Definitions and examples	243
14.2. Limit distributions for extreme values	246
14.3. Relationships to other types of convergence	249
14.4. Convergence conditions for sequences of distributions	252
14.5. Sequences of distributions on $\overline{\mathbb{R}}$	254
14.6. Relative sequential compactness	255
14.7. The Continuity Theorem	260
14.8. Scaling and centering of sequences of distributions	263
14.9. Characterization of moment generating functions	266
14.10. Characterization of characteristic functions	269
Chapter 15. Distributional Limit Theorems for Partial Sums	271
15.1. Infinite series of independent random variables	272
15.2. The Law of Large Numbers revisited	273
15.3. The Classical Central Limit Theorem	275

15.4. The general setting for iid sequences	277
15.5. † Large deviations	280
15.6. † Local limit theorems	282
Chapter 16. Infinitely Divisible Distributions as Limits	289
16.1. Compound Poisson distributions	289
16.2. Infinitely divisible distributions on \mathbb{R}	293
16.3. Levy-Khinchin representations	295
16.4. Infinitely divisible distributions on \mathbb{R}^+	300
16.5. Extension to $\overline{\mathbb{R}}^+$	302
16.6. The triangular array problem: introduction	303
16.7. Iid triangular arrays	306
16.8. Symmetric and nonnegative triangular arrays	310
16.9. † General triangular arrays	313
Chapter 17. Stable Distributions as Limits	323
17.1. Regular variation	324
17.2. The stable distributions	326
17.3. † Domains of attraction	332
17.4. ‡ Domains of strict attraction	342
Chapter 18. Convergence in Distribution on Polish Spaces	347
18.1. Polish spaces	347
18.2. Definition of and criteria for convergence	352
18.3. Relative sequential compactness	355
18.4. Uniform tightness and the Prohorov Theorem	357
18.5. Convergence in product spaces	358
18.6. The Continuity Theorem for \mathbb{R}^d	361
18.7. † The Prohorov metric	363
Chapter 19. The Invariance Principle and Brownian Motion	367
19.1. Certain sequences of distributions on $\mathbf{C}[0,1]$	368
19.2. The existence of and convergence to Wiener measure	371
19.3. Some measurable functionals on $\mathbf{C}[0,1]$	374
19.4. Brownian motion on $[0, \infty)$	379

19.5. Filtrations and stopping times	382
19.6. Brownian motion, filtrations, and stopping times	385
19.7. ‡ Characterization of Brownian motion	389
19.8. † Law of the Iterated Logarithm	390

Part 4. Conditioning

Chapter 20. Spaces of Random Variables	395
20.1. Hilbert spaces	395
20.2. The Hilbert space $L_2(\Omega, \mathbf{F}, \mathbf{P})$	397
20.3. The metric space $L_1(\Omega, \mathcal{F}, P)$	401
20.4. † Best linear estimator	402
Chapter 21. Conditional Probabilities	403
21.1. The construction of conditional probabilities	403
21.2. Conditional distributions	412
21.3. Conditional densities	416
21.4. Existence and uniqueness of conditional distributions	417
21.5. Conditional independence	422
21.6. ‡ Conditional distributions of normal random vectors	427
Chapter 22. Construction of Random Sequences	429
22.1. The basic result	429
22.2. Construction of exchangeable sequences	433
22.3. Construction of Markov sequences	436
22.4. Polya urns	437
22.5. † Coupon collecting	439
Chapter 23. Conditional Expectations	443
23.1. Definition of conditional expectation	443
23.2. Conditional versions of unconditional theorems	448
23.3. Formulas for conditional expectations	451
23.4. Conditional variance	453

Part 5. Random Sequences

Chapter 24. Martingales	459
24.1. Basic definitions	459
24.2. Examples	461
24.3. Doob decomposition	465
24.4. Transformations of submartingales	466
24.5. Another transformation: optional sampling	467
24.6. Applications of optional sampling	473
24.7. Inequalities and convergence results	477
24.8. † Optimal strategy in Red and Black	484
Chapter 25. Renewal Sequences	489
25.1. Basic criterion	490
25.2. Renewal measures and potential measures	491
25.3. Examples	494
25.4. Renewal theory: a first step	497
25.5. Delayed renewal sequences	499
25.6. The Renewal Theorem	502
25.7. † Applications to random walks	506
Chapter 26. Time-homogeneous Markov Sequences	511
26.1. Transition operators and discrete generators	511
26.2. Examples	515
26.3. Martingales and the strong Markov property	520
26.4. Hitting times and return times	522
26.5. Renewal theory and Markov sequences	525
26.6. Irreducible Markov sequences	527
26.7. Equilibrium distributions	528
Chapter 27. Exchangeable Sequences	533
27.1. Finite exchangeable sequences	533
27.2. Infinite exchangeable sequences	539
27.3. Posterior distributions	542
27.4. † Generalization to Borel spaces	544

27.5. ‡ Ferguson distributions and Blackwell-MacQueen urns	550
Chapter 28. Stationary Sequences	553
28.1. Definitions	553
28.2. Notation	555
28.3. Examples	556
28.4. The Birkhoff Ergodic Theorem	558
28.5. Ergodicity	561
28.6. † The Kingman-Liggett Subadditive Ergodic Theorem	564
28.7. ‡ Spectral analysis of stationary sequences	571
Part 6. Stochastic Processes	
Chapter 29. Point Processes	581
29.1. Point processes as random Radon measures	581
29.2. Intensity measures	586
29.3. Poisson point processes	587
29.4. Examples of Poisson point processes.	589
29.5. † Probability generating functionals	594
29.6. ‡ Operations on point processes	597
29.7. ‡ Convergence in distribution for point processes	598
Chapter 30. Lévy Processes	601
30.1. Measurable spaces of right-continuous functions	601
30.2. Definition of Levy process	602
30.3. Construction of Lévy processes	605
30.4. Filtrations and stopping times	610
30.5. † Subordination	611
30.6. ‡ Local-time processes and regenerative subsets of $[0, \infty)$	612
30.7. ‡ Sample function properties of subordinators	618
Chapter 31. Introduction to Markov Processes	621
31.1. Cadlag space	621
31.2. Markov, strong Markov, and Feller processes	622
31.3. Infinitesimal generators	628
31.4. The martingale problem	629

31.5. Pure-jump Markov processes: bounded rates	632
31.6. Pure-jump Markov processes: unbounded rates	636
31.7. † Renewal theory for pure-jump Markov processes	639
Chapter 32. Interacting particle systems	641
32.1. Configuration spaces and infinitesimal generators	641
32.2. The universal coupling	644
32.3. Examples	651
32.4. Equilibrium distributions	655
32.5. Systems with attractive infinitesimal generators	657
Chapter 33. Diffusions and Stochastic Calculus	661
33.1. Stochastic difference equations	661
33.2. The Itô integral	663
33.3. Stochastic differentials and the Itô Lemma	668
33.4. Autonomous stochastic differential equations	672
33.5. Generators and the Dirichlet problem	679
33.6. Diffusions in higher dimensions	682
Part 7. Appendices	
Appendix A. Notation and Usage of Terms	687
A.1. Symbols	687
A.2. Usage	690
A.3. Exercises on subtle distinctions	692
Appendix B. Metric Spaces	693
B.1. Definition	693
B.2. Sequences	694
B.3. Continuous functions	695
B.4. Important metric spaces	695
Appendix C. Topological Spaces	697
C.1. Concepts	697
C.2. Compactification	699
C.3. Product topologies	700
C.4. Relative topology	700

C.5. Limits and continuous functions701
Appendix D. Riemann-Stieltjes Integration703
D.1. The Riemann-Stieltjes integral703
D.2. Relation to the Riemann integral	705
D.3. Change of variables706
D.4. Integration by parts	707
D.5. Improper Riemann-Stieltjes integrals708
Appendix E. Taylor Approximations, C-Valued Logarithms	709
E.1. Some inequalities based on the Taylor formula709
E.2. Complex exponentials and logarithms	711
E.3. Approximations of general C-valued functions	714
Appendix F. Bibliography	715
Appendix G. Comments and Credits	723
Index73 7