

# Homological Algebra

S. I. Gelfand, Yu. I. Manin

## Contents

|   |    |
|---|----|
| Introduction . . . . .  | 4  |
| Chapter 1. Complexes and Cohomology . . . . .                   | 8  |
| §1. Complexes and the Exact Sequence . . . . .                  | 8  |
| §2. Standard Complexes in Algebra and in Geometry . . . . .     | 9  |
| §3. Spectral Sequence . . . . .                                 | 17 |
| Bibliographic Hints . . . . .                                   | 21 |
| Chapter 2. The Language of Categories . . . . .                 | 22 |
| §1. Categories and Functors . . . . .                           | 22 |
| §2. Additive and Abelian Categories . . . . .                   | 35 |
| §3. Functors in Abelian Categories . . . . .                    | 42 |
| §4. Classical Derived Functors . . . . .                        | 47 |
| Bibliographic Hints . . . . .                                   | 52 |
| Chapter 3. Homology Groups in Algebra and in Geometry . . . . . | 52 |
| §1. Small Dimensions . . . . .                                  | 52 |
| §2. Obstructions, Torsors, Characteristic Classes . . . . .     | 56 |
| §3. Cyclic (Co)Homology . . . . .                               | 60 |
| §4. Non-Commutative Differential Geometry . . . . .             | 67 |
| §5. (Co)Homology of Discrete Groups . . . . .                   | 71 |
| §6. Generalities on Lie Algebra Cohomology . . . . .            | 76 |
| §7. Continuous Cohomology of Lie Groups . . . . .               | 77 |
| §8. Cohomology of Infinite-Dimensional Lie Algebras . . . . .   | 81 |
| Bibliographic Hints . . . . .                                   | 85 |

|  |         |
|--|---------|
| Chapter 4. Derived Categories and Derived Functors . . . . .                               | 86      |
| §1. Definition of the Derived Category . . . . .   | 86      |
| §2. Derived Category as the Localization<br>of Homotopic Category . . . . .                | 92      |
| §3. Structure of the Derived Category . . . . .  | 97      |
| §4. Derived Functors . . . . .   | 102     |
| §5. Sheaf Cohomology . . . . .   | 110     |
| Bibliographic Hints . . . . .  | 120     |
| <br>Chapter 5. Triangulated Categories . . . . .   | <br>121 |
| §1. Main Notions . . . . .   | 121     |
| §2. Examples . . . . .   | 128     |
| §3. Cores . . . . .  | 133     |
| Bibliographic Hints . . . . .  | 139     |
| <br>Chapter 6. Mixed Hodge Structures . . . . .  | <br>140 |
| §0. Introduction . . . . .   | 140     |
| §1. The Category of Hodge Structures . . . . .   | 142     |
| §2. Mixed Hodge Structures on Cohomology . . . . .<br>with Constant Coefficients . . . . . | 145     |
| §3. Hodge Structures on Homotopic Invariants . . . . .                                     | 148     |
| §4. Hodge-Deligne Complexes . . . . .  | 153     |
| §5. Hodge-Deligne Complexes for Singular<br>and Simplicial Varieties . . . . .             | 155     |
| §6. Hodge-Beilinson Complexes and Derived Categories<br>of Hodge Structures . . . . .      | 157     |
| §7. Variations of Hodge Structures . . . . .   | 159     |
| Bibliographic Hints . . . . .  | 162     |
| <br>Chapter 7. Perverse Sheaves . . . . .  | <br>163 |
| §1. Perverse Sheaves . . . . .   | 163     |
| §2. Glueing . . . . .  | 168     |
| Bibliographic Hints . . . . .  | 172     |
| <br>Chapter 8. $\mathcal{D}$ -Modules . . . . .  | <br>173 |
| §0. Introduction . . . . .   | 173     |
| §1. The Weyl Algebra . . . . .   | 175     |
| §2. Algebraic $\mathcal{D}$ -Modules . . . . .   | 182     |
| §3. Inverse Image . . . . .  | 188     |
| §4. Direct Image . . . . .   | 190     |
| §5. Holonomic Modules . . . . .  | 195     |
| §6. Regular Connections . . . . .  | 202     |

§7. D-Modules with Regular Singularities . . . . . 205

§8. Equivalence of Categories (Riemann-Hilbert Correspondence) . . . 208

Bibliographic Hints . . . . . 210

References . . . . . 2 11

Author Index . . . . . 217

Subject Index . . . . . 219