

# Preface

This book is based on a two-semester course in “The Mathematical Methods of Physics” which I have given in the mathematics department of the University of Illinois in recent years. The audience has consisted primarily of physicists, engineers, and other natural scientists in their first or second year of graduate study. Knowledge of the theory of functions of real and complex variables is assumed.

The subject matter has been shaped by the needs of the students and by my own experience. In many cases students who do not major in mathematics have room in their schedules for only one or two mathematics courses. The purpose of this book, therefore, is to provide the student with some heavy artillery in several fields of mathematics, in confidence that targets for these weapons will be amply provided by the student’s own special field of interest. Naturally, in such an attempt, something must be sacrificed, and I have regarded as most expendable discussions of physical applications of the material being presented.

Again, in the short space allotted to each subject there is little chance to develop the theory beyond fundamentals. Thus in each chapter I have gone straight to (what I regard as) the heart of the matter, developing a subject just far enough so that applications can easily be made by the student himself. The exercises at the end of each chapter, along with their solutions at the back of the book, afford some further opportunities for using the theoretical apparatus.

The material herein is, for the most part, classical. The bibliographical references, particularly to journal articles, are given not so much to provide a jumping-off point for further research as to give the reader a

feeling for the chronological development of these subjects and for the names of the men who created them.

Finally, I have, where possible, tried to say something about numerical methods for computing the solutions of various kinds of problems. These discussions, while brief, are oriented toward electronic computers and are intended to help bridge the gap between the “there exists” of a pure mathematician and the “find it to three decimal places” of an engineer.

I am indebted to Professor L. A. Rubel for permission to publish Theorem 7 of Chapter 3 here for the first time and to Professor R. P. Jerrard for some of the exercises in Chapter 7. To the well-known volume of Courant and Hilbert I owe the intriguing notion that, even in an age of specialization, it may be possible for physicists and mathematicians to understand each other.

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