REALITY RULES: I

Picturing the World in Mathematics —The Fundamentals—

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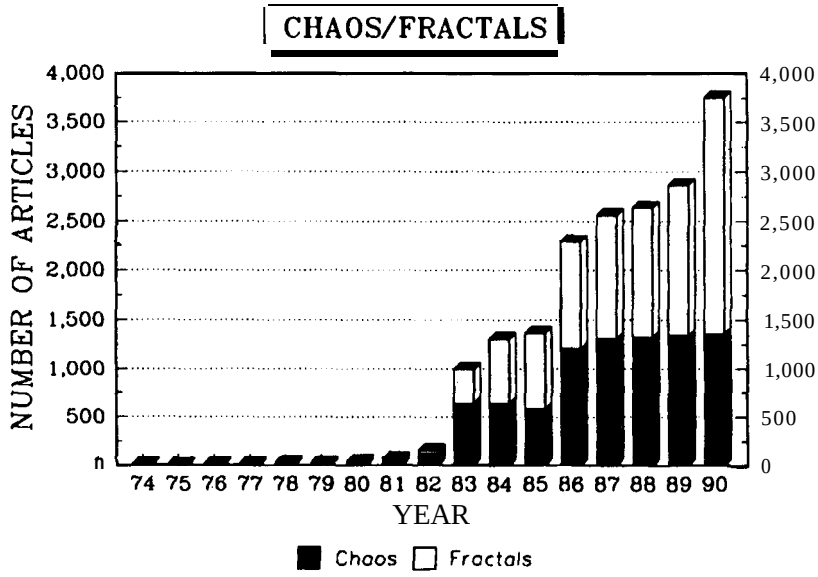
PREFACE

Mathematical modeling is about rules-the rules of reality. What distinguishes a mathematical model from, say, a poem, a song, a portrait or any other kind of "model," is that the mathematical model is an image or picture of reality painted with logical symbols instead of with words, sounds or watercolors. These symbols are then strung together in accordance with a set of rules expressed in a special language, the language of mathematics. A large part of the story told in the 800 pages or so comprising the two volumes of this work is about the grammar of this language. But a piece of the real world encoded into a set of mathematical rules (i.e., a model) is itself an abstraction drawn from the deeper realm of "the real thing." Based as it is upon a choice of what to observe and what to ignore, the real-world starting point of any mathematical model must necessarily throw away aspects of this "real thing" deemed irrelevant for the purposes of the model. So when trying to fathom the meaning of the title of this volume, I invite the reader to regard the word "rule" as either a noun or a verb-or even to switch back and forth between the two-according to taste.

The book you now hold in your hands started its life as a simple revision of my 1989 text-reference *Alternate Realities*. But like Topsy it just sort of grew, until it reached the point where it would have been a misnomer, if not a miscarriage of justice, to call the resulting book a "second" or "revised" or "updated" or even a "new" edition of that earlier work. And, in fact, the project grew to such an extent that sensibility and practical publishing concerns dictated a. splitting of the work into two independent, yet complementary, volumes. But before giving an account of the two halves of my message, let me first offer a. few words of explanation as to why a three-year old book needed updating in the first place.

Alternate Realities was a mathematical-modeling text devoted to bringing the tools of modern dynamical system theory into the classroom. As such, the focus of the book was on things like chaos, linear system theory,

cellular automata, evolutionary game theory, q-analysis and the like. While the book itself was published in early 1989, the actual writing took place during late-1986. As a result, most of the material was based on what was current in the research literature *circa* 1985. In the intervening years research interest and results in dynamical system theory has been nothing short of explosive. The chart below gives some indication of the magnitude of this exponential growth of published research in just the fields of chaos and fractals. And work in the other topical areas addressed in *Alternate Realities* has certainly been no less intense. Hence, the call from both readers and my editor for an update.



So what are you getting in the two halves of this book that's new besides some polishing of the reference lists to a little brighter shine and a tidying-up of typos and other literary infelicities in *Alternate Realities*? Briefly, here are the main attractions:

- *New chapter and* sections-Recognizing the fact that for a modern mathematician a computer program has become a legitimate answer to a mathematical question, *The Frontier* contains an entirely new chapter (Chapter 9) on computation and complexity. In particular, this chapter takes up the question of what can actually be done by way of formalizing real-world phenomena within the confines of a computer program, i.e., a set of rules. This chapter offers the student and researcher a point of contact with a host of matters of current intellectual concern running the gamut from Gödel's Incompleteness Theorem and its connection to work in artificial intelligence to the problem of NP-completeness and the complexity of numerical algorithms.

In addition to the new chapter, a number of new sections have been added to the chapters originally constituting *Alternate Realities*. These sections introduce topical areas like artificial life that were only touched on briefly in that earlier work, or treat entirely new areas of concern like the relation between chaos and stock-price fluctuations that were yet to be born in 1985.

- *Exercises*—One of the main comments (and complaints) I heard from readers and users of *Alternate Realities* was that the problems were too hard. Too hard, anyway, to be used as drill exercises in a classroom setting. To remedy this defect, I have added Exercises of the drill-type at the end of almost every section in every chapter of the book (except for the final chapter, which is more a meditation on the philosophy of modeling than on the mathematics). These Exercises make the book much easier for instructors to use as a mathematical modeling text in the classroom, as well as provide drill problems for those using the book for self-study.

- *Examples*- As a further effort toward making the material accessible to students and researchers alike, many new examples have been sprinkled throughout the book at strategic locations. Not only do these examples introduce new applications of the theoretical results, they also serve to illustrate exactly how to use the theory in a diverse array of realistic situations.

- Discussion *Questions* and Problems-Several new Discussion Questions have been added to each chapter, both to extend the earlier material and to bring forth ideas nonexistent a few years ago. Moreover, the Problems sections have been augmented with new results from the research literature that for one reason or another didn't fit into the mainstream of the text.

- *Solutions Manual*—From an instructor's point of view, it's always a bit awkward, not to mention annoying, trying to teach from a book containing problems that you yourself can't solve! And I must confess that the problems in *Alternate Realities* were by and large pretty hard, most of them having been taken directly from the primary research literature. So to ease the pain of grappling with these research-type questions, I have prepared a Solutions Manual for the problems appearing in each chapter's *Problems* section. This manual is available at no charge from the publisher. To obtain a copy, ask your Wiley sales representative or write directly to Wiley-Interscience, 605 Third Avenue, New York, NY 10158-0012.

So there you have it. More than 300 pages of new points of contact between the worlds of nature and mathematics-in itself material constituting a fair-sized book. I think that after looking over this new material, readers of *Alternate Realities* and librarians will both agree that a new book was indeed in order. Now let me talk for a. moment about the intended audience for this book and how it might be used.

One of the questions that always comes up in deciding on a text for a course is the puzzler: What are the prerequisites? Since this is a book on mathematical modeling, it goes without saying that in general terms the prerequisites for using the book are a working *knowledge* of basic undergraduate mathematics. This means that the reader should have been exposed to *and assimilated* the material typically found in one-semester courses in calculus, linear algebra and matrix theory, ordinary differential equations and, perhaps, elementary probability theory. Moreover, s/he should know about the basic vocabulary and techniques of mathematics. So, for example, things like "sets" and "equivalence relations" should be familiar territory, as should the idea of an inductive argument or a proof by contradiction. The emphasis here, of course, is on the phrase "working knowledge." To illustrate, it is definitely not sufficient to have had a course in matrix theory and have just heard of the Cayley-Hamilton Theorem. You must actually know what the theorem says and how to use it. Or, at the bare minimum, you should at least know where to go and look up the result. This is what I mean when I say the prerequisites for accessing this book are a working knowledge of basic undergraduate mathematics.

Now let's turn to how the book might be used as a text in a course in mathematical modeling. It's manifestly evident, I think, that there's far more material in these two volumes than can be comfortably addressed in even a one-year course, let alone in a single semester. Hence, one of the prime motivations for splitting the book into two pieces. Since what distinguishes *modern* mathematical modeling from its classical counterpart is its emphasis upon dynamics and nonlinearity, *The Fundamentals* contains the essentials of these matters. By way of illustration, let me briefly indicate how I have made use of this material in my own courses.

First of all, I always include the material from Chapter 1 as it sets the general framework for just about every type of mathematical modeling undertaking. I sometimes follow Chapter 1 with selected material from Chapter 8 on q-analysis. For some reason students seem to find it easier to grasp what's going on in the modeling game by starting with the context of sets and binary relations rather than jumping immediately into the thicket of dynamical systems and its daunting terminology and definitions. Moreover, the generality of the q-analysis idea lends itself to a number of interesting examples in art, literature and life that lie outside the domain of the "hard" sciences. Since just about everyone nowadays wants to know about chaos, my lectures usually continue with a. selection of material from Chapters 2, 3 and 4. The second chapter contains the necessary background for dealing with dynamical systems, which is then used in Chapters 3 and 4 within the specific settings of cellular automata and chaotic processes. If there is still time remaining in the term, I conclude the course with a discussion of some of the more general philosophical issues linking science and other reality-

generating mechanisms, as outlined in the book's final chapter. Since this program involves bits and pieces from *The Frontier,* it would be cruel and inhuman, not to mention imprudent, to close this Preface without saying a word or two about the second half of the book.

As I've already mentioned, the sheer size of this "magnum opus" dictated its division into two volumes. But unlike many multi-volume efforts, this split was not made on the basis of introductory versus advanced material and/or techniques. Quite to the contrary, in fact, as in putting this book together I tried valiantly to ensure that the difficulty level was as uniform as possible throughout. So the line of demarcation between the two volumes lies in a very different direction.

The Fundamentals contains the material that I feel is essential for anyone to consider himself or herself a player in the game of "modern" mathematical modeling. Just as it's difficult to conceive of writing a book without knowing the alphabet, there are certain concepts and results that anyone who wants to get involved with modeling must have at his or her disposal. *The Fundamentals* provides an account of this irreducible minimum of basic knowledge, material that can form the basis for a one-semester initial encounter with mathematical modeling.

For those already familiar with the essentials of system modeling, *The Frontier* introduces a. number of applica.tion areas and/or associated techniques of modeling that complement the ideas presented in *The Fundamentals.* Chapter 5 shows how dynamicaI system theory and concepts from game theory can be brought together to shed new light on problems of population biology and ecology. This chapter also gives a mathematical account of the controversial problem of sociobiology, as well as taking a long look at the emergence of cooperative behavior in a. world of egoists.

In Chapter 6 the notion of a control system is introduced within the cozy confines of 1inear processes. Of special importance in this chapter are the ideas of reachability and observability, concepts that form the basis for what we can hope to know about a system from measurements made upon it, as well as the degree to which we can alter a system's behavior by feeding inputs into it from the outside. By restricting the setting to linear structures, these notions are used to show how to construct "good" models directly from observed data. The chapter concludes with an indication of how these ideas can be extended to more general types of nonlinear processes.

If a system's behavior can be altered by applying inputs from the outside, how should those inputs be chosen so as to maximize or minimize some measure of system performance? This question is the central theme of Chapter 7. Following an introduction to the two principal techniques for calculating optimal control policies-the Minimum Principle and dynamic programming-the chapter presents an extended consideration of a novel class of adaptive control processes motivated by considerations aris-

ing in cellular biology. These so-called "(M, R)-systems" are then used to formalize ideas of self-reference and anticipation in application areas like manufacturing operations and input/output economics.

Shifting the emphasis from dynamics and continuity to statics and discrete combinatorial structures, Chapter 8 addresses the ways in which patterns in art, literature and other areas of human endeavor outside the natural sciences can be formulated in meaningful mathematical terms. Employing the ideas of q-analysis, which is a kind of extension of classical graph theory, this chapter develops a number of applications ranging from M. C. Escher engravings to humor and pathos in literature. These applications show how mathematical structures lurk at the heart of such reality-generation mechanisms heretofore thought to be beyond the bounds of mathematical analysis.

The thread running through every chapter of both volumes of *Reality Rules* is the idea that a mathematical model is a particular way of expressing a rule. In today's world, this rule is formulated in terms of a computer program more often than not. Chapter 9 is devoted to showing why there is no difference that matters between a computer program, a dynamical system and a deductive logical system. Exploiting the equivalence between these three seemingly distinct ways of formalizing a real-world situation, the chapter shows how the results of Turing, Gödel and Chaitin lead us to consider the ultimate limitations of the scientific method as a procedure for telling us the way the world works.

Having spent nine chapters looking at the scientific way of getting at the scheme of things, the book's final chapter turns a bit philosophical and examines the degree to which the answers provided by science are in any way superior to those provided by other reality-generation procedures like religion, mysticism, music, art or literature. Following a discussion of science versus pseudoscience, the chapter concludes with the message that there are many realities, of which the scientific variety have no special claim to intrinsic superiority.

Taken together, it's my hope that the material of the two volumes making up *Reality Rules* will serve as both a text and a reference that students and researchers alike will be able to turn to as a source of inspiration and information as they make their way through the ever-shifting quicksand and minefields of the complex, weird and wonderful world we all inhabit.

JLC
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