Nonlinear Dynamics

A Two-way Trip from Physics to Math

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Contents

Acknowledgments xv

Preface xvii

1 Nonlinear dynamics in nature 1
1.1 Hiking among rabbits 2
1.2 Turbulence 3
1.3 Bénard instability 6
1.4 Dynamics of a modulated laser 10
1.5 Tearing of a plasma sheet 13
1.6 Summary 16

2 Linear dynamics 19
2.1 Introduction 19
2.2 Why linear dynamics? 20
2.3 Linear flows 20
2.3.1 Autonomous flows 21
2.3.1.1 Autonomous homogeneous flows 21
2.3.1.2 Autonomous inhomogeneous flows 26
2.3.1.3 Singular points in bi-dimensional flows 27
2.3.2 Forced flows 29
2.3.2.1 Additively forced flows 30
2.3.2.2 Parametrically forced flows 31
2.3.2.3 Floquet's theorem 33
2.3.2.4 Oscillation theorem-Liapunov-Haupt 35
2.3.2.5 Arnold tongues 37
2.4 Summary 38
2.5 Additional exercise 38

3 Nonlinear examples 39
3.1 Preliminary comments 39
3.2 A model for the CO_2 laser 39
3.2.1 The model 39
3.2.2 Dynamics of the laser 42
3.3 Duffing oscillator 44

3.3.1 The model 44
3.3.2 Dynamics in the unforced model 44
3.4 The Lorenz equations 45
3.4.1 The model 46
3.4.1.1 Properties 48
3.4.2 Simple dynamics in the Lorenz model 49
3.5 Summary 49
3.6 Additional exercises 51

4 Elements of the description 53
4.1 Introduction 53
4.2 Basic elements 54
4.2.1 Phase space 54
4.2.2 Flow 55
4.2.3 Invariants 56
4.2.3.1 Orbits 56
4.2.4 Attractors 58
4.2.4.1 Attracting sets 58
4.2.4.2 Attractors 58
4.2.4.3 Basin of attraction 59
4.2.4.4 Boundaries of the basin of attraction 60
4.2.5 Trapping regions 60
4.2.6 Stable and unstable sets 61
4.3 Poincare sections 62
4.3.1 Stroboscopic section 62
4.3.2 Transverse section 63
4.3.3 Poincart sections and Poincart first-return maps 64
4.3.3.1 Global Poincare section 64
4.3.3.2 Local Poincare section 65
4.4 Maps and dynamics 66
4.4.1 Properties of maps 67
4.4.1.1 Conjugated maps 67
4.5 Parameter dependence 68
4.5.1 Families of flows and maps 68
4.6 Summary 69
4.7 Additional exercise 69

5 Elementary stability theory 71
5.1 Introduction 71
5.2 Fixed point stability 72
5.2.1 Liapunov stability criteria 73
5.3 The validity of the linearization procedure 76
5.4 Maps and periodic orbits 79
5.4.1 Maps 79
5.4.2 Periodic orbits of flows, Floquet stability theory 80

5.5 Structural stability 82
5.5.1 Orbital equivalence 82
5.5.2 Structural equivalence 83
5.6 Summary 84
5.7 Additional exercise 84

6 Bi-dimensional flows 85
6.1 Limit sets 85
6.2 Transverse sections and sequences 87
6.3 Poincaré–Bendixson theorem 88
6.4 Structural stability 92
6.5 Summary 93

7 Bifurcations 95
7.1 The bifurcation programme 95
7.1.1 Families of flows and maps 95
7.1.2 Local and global bifurcations 96
7.1.3 Local bifurcations: the programme 96
7.2 Equivalence between flows 97
7.3 Conditions for fixed point bifurcations 98
7.3.1 Unfolding of a bifurcation loo
7.4 Reduction to the centre manifold 101
7.4.1 Adiabatic elimination of fast-decaying variables 101
7.4.2 Approximation to the centre manifold 102
7.4.3 Centre manifold for maps 105
7.4.4 Bifurcations 106
7.5 Normal forms 107
7.5.1 Example: Jordan block 107
7.5.2 The general case 109
7.5.3 Normal forms for maps 110
7.5.4 Resonances 112
7.5.4.1 Small denominators 113
7.6 Simplest local bifurcations of fixed points 113
7.6.1 Saddle-node bifurcation (flows) 114
7.6.2 Pitchfork and transcritical bifurcations 114
7.6.2.1 Pitchfork bifurcation 114
7.6.2.2 Transcritical bifurcation 115
7.6.2.3 Structural stability 116
7.6.3 Hopf bifurcation (flows) 118
7.6.4 General features of fixed point bifurcations 120
7.7 Bifurcations of maps and periodic orbits 120
7.7.1 Saddle-node bifurcation 120
7.7.2 Flip bifurcation 122
7.7.3 Hopf bifurcation (maps) 123
7.7.3.1 Non-resonant Hopf bifurcation 125

7.7.4 Hopf bifurcation at a weak resonance 126
7.7.4.1 Hopf bifurcation at a 1/3 resonance 127
7.7.4.2 Hopf bifurcation at a 1/4 resonance 128
7.8 Remarks 129
7.9 Summary 131
7.10 Additional exercises 131

8 Numerical experiments 135
8.1 Period-doubling cascades 136
8.1.1 Period doubling in the CO_2 laser 136
8.1.1.1 Manifold organization 139
8.1.2 Scaling ratios 142
8.1.3 Period halving: the inverse (noisy) cascade 144
8.2 Torus break-up 145
8.2.1 Curry and Yorke's experiment 145
8.2.2 Analysis of the map 147
8.3 Homoclinic explosions in the Lorenz system 150
8.4 Chaos and other phenomena 153
8.4.1 Crisis 153
8.4.2 Chaotic basin boundaries and chaotic transients 156
8.4.3 Chaotic scattering 158
8.5 Summary 159

9 Global bifurcations: I **161**
9.1 Transverse homoclinic orbits 161
9.1.1 Linear approximation 162
9.2 Homoclinic tangencies 164
9.3 Homoclinic tangles and horseshoes 165
9.4 Heteroclinic tangles 166
9.5 Summary 169

10 Horseshoes **171**
10.1 The invariant set 171
10.1.1 A horseshoe prototype 175
10.2 Cantor sets 176
10.3 Symbolic dynamics 177
10.3.1 Restriction of the map to the invariant set 178
10.3.2 Periodic points of F^n 179
10.3.3 Orbit dense in A 179
10.3.4 Stable and unstable foliations 180
10.3.5 Sensitivity to initial conditions 180
10.4 Horseshoes and attractors 181
10.4.1 Smale's theorem 182
10.5 Hyperbolicity 183
10.6 Structural stability 184

10.7 Summary 185
10.8 Additional exercises 185

11 One-dimensional maps 187
11.1 Unimodal maps of the interval 188
11.1.1 Itineraries 190
11.1.1.1 Order on the itineraries 191
11.1.1.2 Reasonable itineraries 192
11.1.1.3 Reasonable itineraries that are represented by some point 193
11.1.1.4 Some examples 198
11.1.1.5 The conclusion 200
11.2 Elementary kneading theory 201
11.2.1 Kneading of periodic orbits 201
11.2.1.1 Orbit implication 202
11.3 Parametric families of unimodal maps 204
11.4 Summary 207

12 Topological structure of three-dimensional flows 209
12.1 Introduction 209
12.2 Forced oscillators and two-dimensional maps 210
12.3 Topological invariants 214
12.3.1 The Conway polynomial and the linking number 215
12.3.2 Braids 218
12.3.3 Relative rotation rates 220
12.4 Orbits that imply chaos 221
12.5 Horseshoe formation 223
12.5.1 Hénon map and the logistic map 223
12.5.2 Orbit organization 225
12.5.3 Subtleties of orbit organization 227
12.5.4 Suspensions 227
12.5.5 Templates 229
12.6 Topological classification of strange attractors 230
12.7 Summary 232

13 The dynamics behind data 233
13.1 Introduction and motivation 233
13.2 Characterization of chaotic time-series 234
13.2.1 Imbeddings 235
13.2.2 Dimensions 238
13.2.3 Liapunov exponents 241
13.2.4 Entropy 242
13.2.4.1 Probability entropy 242
13.2.4.2 Topological entropy 244
13.3 Is this data set chaotic? 244

13.3.1 The well known enemies 245
13.3.2 Testing the tests 247
13.3.2.1 Testing with surrogate data 248
13.3.3 Topological tests: how chaotic is our data set? 250
13.4 Summary 251

14 Perturbative methods: averaging 253
14.1 The perturbative problem 254
14.1.1 The problem in standard form 254
14.1.2 A first view of averaging 254
14.2 A non-trivial example 256
14.2.1 Rate equations for the laser with injected signal 256
14.2.2 Averaged equations 257
14.2.3 Fixed points and bifurcations 260
14.3 The periodic case 263
14.3.1 Bifurcations 268
14.4 Perturbation of a Hamiltonian problem 268
14.4.1 General position 270
14.4.2 Resonant layer 271
14.4.3 Perturbation of fixed points 273
14.5 Perturbation of homoclinic loops 274
14.6 Hopf bifurcation at a strong resonance 279
14.6.1 The 1/3 resonance 281
14.6.2 The 1/4 resonance 283
14.6.2.1 Case $|A| > 1$ 284
14.6.2.2 Case $|A| < 1$ 284
14.7 Summary 286
14.8 Additional exercise 287

15 Bifurcations and symmetries 289
15.1 Introduction 289
15.2 Symmetries and groups 290
15.3 Intensity patterns in lasers 291
15.4 Symmetries and bifurcation theory: the tricks 293
15.4.1 Symmetries and linear theory 294
15.4.2 Invariant theory 296
15.5 The treats 297
15.6 Revisiting the laser 300
15.7 Imperfect symmetries 302
15.8 Summary 303
15.9 Additional exercises 304

16 Global bifurcations: II 307
16.1 Introduction 307
16.2 Homoclinic orbits of maps and flows 307

16.2.1 Manifold organization around a homoclinic orbit 308
16.2.2 Manifold organization around periodic orbits of flows 309
16.2.2.1 Intersection of manifolds 310
16.3 Homoclinic orbit in two-dimensional flows (to a fixed point) 311
16.4 Homoclinic orbit in three-dimensional flows (to a fixed point) 314
16.4.1 Homoclinic orbit with real eigenvalues 314
16.4.1 .1 Parameter versus period plots 317
16.4.2 Homoclinic explosions: homoclinic orbit with symmetry 318
16.4.2.1 Local and global maps: a geometric approach. 319
16.4.3 Homoclinic orbit with complex eigenvalues 322
16.4.3.1 Parameter versus period plots in the Šilnikov phenomenon 324
16.4.3.2 The Šilnikov's scenario and horseshoes 326
16.5 The quest for homoclinic connections 327
16.5.1 Heteroclinic cycles in symmetric systems 328
16.5.2 Homoclinic orbits in the laser 329
16.5.2.1 Local and global maps 330
16.5.2.2 Homoclinic orbits and the Silnikov phenomenon 332
16.6 Summary 333

References 335

Index 343

Preface

About the book

The aim of this book is to render available to readers the tools that nonlinear dynamics provides for the exploration of new problems in all fields of physics. In research we deal with open problems: problems for which, at the beginning, we have no solutions but at most a set of hunches, feelings and guesses based on our previous experience with other problems. From there, we work our path to the solutions (though we do not always succeed). In finding our way we take what seems to us the ***most natural*** approach just as we would bush-walk in the forest avoiding as much as possible the difficult paths in our hike towards an interesting place.

We will follow in the presentation the same procedure that we follow when doing research, i.e., we begin with the problem and find one path to the solution, we work inductively proposing new paths, checking them and redrawing our route according to the experience we gain in successive efforts.

We will avoid the temptation of selecting the problems according to the tools we possess. Rather, we prefer to construct the tools along with the problems. We sustain the idea that our tools (theories) and our problems evolve hand in hand. The best pages of physics have been written in this way. Consider for example the pairs calculus-mechanics (Newton) or Hilbert spaces-quantum theory (Von Neumann) and, as we will see, nonlinear dynamics-topology (Poincaré).

This is a book to be read with paper and pencil at hand. Our intention is to furnish readers with enough knowledge to be able to do research in nonlinear dynamics after having read the book (or better, ***while*** they are reading the book). We prefer to convey the key ideas within their mathematical framework rather than doing lengthy demonstrations. Therefore, the calculations around the results presented in the book are usually only sketched or left more or less as guided exercises.

At the end of the day, we would like the reader to finish this book with the feeling (or certainty) that, given enough time, she/he would have come up with the same answers to the problems as those we have shown (well . . . perhaps just better answers). After all ***the answers are dictated by the problems,*** the two of them evolve in interaction, and our task is to read them from nature.

About nonlinear dynamics

Nonlinear dynamics is a subject at least as old in physics as Newton's mechanics. The dynamics of the planetary system, a primary concern for Newton, Poincart and many others, turns out to be nonlinear in general, but fortunately the simplest examples can be solved exactly (two-body systems are completely integrable). In the context of Hamiltonian mechanics a completely integrable system is 'almost' a linear system in an appropriate set of coordinates (those given by the Hamilton–Jacobi theory for integrable systems [arno89]).

In contrast, three-body problems are also nonlinear, but in general very complex and non-integrable. It was while studying the three-body problem that Poincaré gave a new and important impulse to nonlinear dynamics at the beginning of the 20th century. However, the new physics of the atom (later the nucleus, then quarks, . . .) caught the attention of physicists. There was apparently no use for nonlinear dynamics in quantum mechanics since the latter rests on Hilbert spaces (linear spaces after all). The excess of zeal with quantum mechanics caused the (almost complete) disappearance of nonlinear dynamics from physics and especially from textbooks.

The emergence of computers as new tools for theoretical physics in the late 1950s and early 1960s favoured a comeback of nonlinear dynamics. Numerical simulations made accessible to the intuition of physicists and non-physicists the richness of nonlinear models. The graphical output added an artistic touch.

Commensurate with its earlier neglect, the impact of the phenomenology of nonlinear dynamics rocked the physics community in the early 1980s, to the point that people even talked of a 'new science' [glei87].

Today, we have a calmer perspective. We recognize that many situations can only be described with nonlinear interactions. There is a growing consciousness that the tools, methods and phenomenology of nonlinear dynamics will be increasingly necessary for the study of most subjects in physics and natural science in general.

The present text is far from being a complete guide to nonlinear dynamics but it covers the basic ideas for general systems. Some topics, though important for historical and even practical reasons for physicists, like one-dimensional maps and Hamiltonian mechanics, have not been emphasized, ***on purpose.*** We are certain that if we were to stress these special (singular) cases we would induce the wrong generalization, just as our generation was induced to think that Hamilton-Jacobi theory applied to all systems.

The discussion is presented in most cases having in mind low-dimensional systems, i.e., systems where the spatial aspects behave coherently. In general, spatio-temporal dynamics is known to a lesser degree than low-dimensional dynamics and the authors' knowledge of the subject is correspondingly more limited.