

a cursory manner. Sometimes it will then be necessary later to go back and study an earlier starred section (e.g. § 37 will have to be studied for § 72).

Gödel's two famous incompleteness theorems are reached in Chapter VIII, leaving a lemma to be proved in Chapter X. The author has found it feasible to complete these ten chapters (and sometimes a bit more) in the semester course which he has given along these lines at the University of Wisconsin.

The remaining five chapters can be used to extend such a course to a year course, or as collateral reading to accompany a seminar.

A semester course on recursive functions for students having some prior acquaintance with mathematical logic, or under an instructor with such acquaintance, could start with Part III (Chapter IX). There are other possibilities for selecting material; e.g. much of Part IV can follow directly Part II or even Chapter VII for students primarily interested in mathematical logic.

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PART I

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