Contents

Foreword: MASS at Penn State University	
Preface	xiii
Guide for instructors	xvii
Chapter 1. Elements of group theory	1
Lecture 1. First examples of groups	1
a. Binary operations	1
b. Monoids, semigroups, and groups	3
c. Examples from numbers and multiplication tables	8
Lecture 2. More examples and definitions	11
a. Residues	11
b. Groups and arithmetic	13
c. Subgroups	15
d. Homomorphisms and isomorphisms	18
Lecture 3. First attempts at classification	20
a. Bird's-eye view	20
b. Cyclic groups	22
c. Direct products	25
d. Lagrange's Theorem	29
Lecture 4. Non-abelian groups and factor groups	31
a. The first non-abelian group and permutation groups	31

VI Cor	itents
b. Representations and group actions	34
c. Automorphisms: Inner and outer	- 38
d. Cosets and factor groups	42
Lecture 5. Groups of small order	46
a. Structure of finite groups of various orders	46
b. Back to permutation groups	50
c. Parity and the alternating group	54
Lecture 6. Solvable and nilpotent groups	57
a. Commutators: Perfect, simple, and solvable groups	57
b. Solvable and simple groups among permutation groups	60
c. Solvability of groups and algebraic equations	65
d. Nilpotent groups	65
Chapter 2. Symmetry in the Euclidean world: Groups of	
isometries of planar and spatial objects	69
Lecture 7. Isometries of \mathbb{R}^2 and \mathbb{R}^3	69
a. Groups related to geometric objects	69
b. Symmetries of bodies in \mathbb{R}^2	72
c. Symmetries of bodies in \mathbb{R}^3	77
Lecture 8. Classifying isometries of \mathbb{R}^2	82
a. Isometries of the plane	82
b. Even and odd isometries	83
c. Isometries are determined by three points	85
d. Isometries are products of reflections	87
e. Isometries in \mathbb{R}^3	90
Lecture 9. The isometry group as a semidirect product	91
a. The group structure of $\mathrm{Isom}(\mathbb{R}^2)$	91
b. Isom ⁺ (\mathbb{R}^2) and its subgroups $G_{\mathbf{p}}^+$ and \mathcal{T}	94
c. Internal and external semidirect products	96
d. Examples and properties of semidirect products	98
Lecture 10. Discrete isometry groups in \mathbb{R}^2	100
a. Finite symmetry groups	100
b. Discrete symmetry groups	105
c. Quotient spaces by free and discrete actions	112
Lecture 11. Isometries of \mathbb{R}^3 with fixed points	114
a. Classifying isometries of \mathbb{R}^3	$11\overline{4}$

Contents	vii
b. Isometries of the sphere	117
c. The structure of $SO(3)$	118
d. The structure of $O(3)$ and odd isometries	120
Lecture 12. Finite isometry groups in \mathbb{R}^3	121
a. Finite rotation groups	121
b. Combinatorial possibilities	126
c. A unique group for each combinatorial type	129
Lecture 13. The rest of the story in \mathbb{R}^3	133
a. Regular polyhedra	133
b. Completion of classification of isometries of \mathbb{R}^3	139
Lecture 14. A more algebraic approach	143
a. From synthetic to algebraic: Scalar products	143
b. Convex polytopes	148
c. Regular polytopes	150
Chapter 3. Groups of matrices: Linear algebra and symmetry in various geometries	155
Lecture 15. Euclidean isometries and linear algebra a. Orthogonal matrices and isometries of \mathbb{R}^n	ı 155 155
 a. Orthogonal matrices and isometries of R" b. Eigenvalues, eigenvectors, and diagonalizable magnetic matrices. 	
c. Complexification, complex eigenvectors, and rote	
d. Differing multiplicities and Jordan blocks	163
Lecture 16. Complex matrices and linear representations	ations 165
a. Hermitian product and unitary matrices	165
b. Normal matrices	170
c. Symmetric matrices	173
d. Linear representations of isometries and more	174
Lecture 17. Other geometries	176
a. The projective line	176
b. The projective plane	181
c. The Riemann sphere	185
Lecture 18. Affine and projective transformations	187
a. Review of various geometries	187
b. Affine geometry	190
c. Projective geometry	195

vii

viii	Contents	Contents	<u>ix</u>
Lecture 19. Transformations of the Riemann sphere	197	Chapter 5. From groups to geometric objects and back	269
a. Characterizing fractional linear transformations	197	Lecture 26. The Cayley graph of a group	269
b. Products of circle inversions	199	a. Finitely generated groups	269
c. Conformal transformations	202	b. Finitely presented groups	273
d. Real coefficients and hyperbolic geometry	203	c. Free products	279
Lecture 20. A metric on the hyperbolic plane	204	Lecture 27. Subgroups of free groups via covering spaces	280
a. Ideal objects	204	a. Homotopy types of graphs	280
b. Hyperbolic distance	205	b. Covering maps and spaces	283
c. Isometries of the hyperbolic plane	209	c. Deck transformations and group actions	286
Lecture 21. Solvable and nilpotent linear groups	212	d. Subgroups of free groups are free	289
a. Matrix groups	212	Lecture 28. Polygonal complexes from finite presentations	290
b. Upper-triangular and unipotent groups	214	a. Planar models	290
c. The Heisenberg group	216	b. The fundamental group of a polygonal complex	296
d. The unipotent group is nilpotent	218	Lecture 29. Isometric actions on \mathbb{H}^2	302
Lecture 22. A little Lie theory	221	a. Hyperbolic translations and fundamental domains	302
a. Matrix exponentials	221	b. Existence of free subgroups	307
b. Lie algebras	224	Lecture 30. Factor spaces defined by symmetry groups	311
c. Lie groups	227	a. Surfaces as factor spaces	311
d. Examples	229	b. Modular group and modular surface	315
Chapter 4. Fundamental group: A different kind of grou	_	c. Fuchsian groups	320
associated to geometric objects	р 233	d. Free subgroups in Fuchsian groups	322
•		e. The Heisenberg group and nilmanifolds	325
Lecture 23. Homotopies, paths, and π_1	233	Lecture 31. More about $SL(n,\mathbb{Z})$	327
a. Isometries vs. homeomorphisms	233	a. Generators of $SL(2,\mathbb{Z})$ by algebraic method	327
b. Tori and \mathbb{Z}^2	235	b. The space of lattices	329
c. Paths and loops	238	c. The structure of $SL(n,\mathbb{Z})$	331
d. The fundamental group	241	d. Generators and generating relations for $SL(n, \mathbb{Z})$	333
e. Algebraic topology	246		337
Lecture 24. Computation of π_1 for some examples	246	Chapter 6. Groups at large scale	
a. Homotopy equivalence and contractible spaces	246	Lecture 32. Introduction to large scale properties	337
b. The fundamental group of the circle	251	a. Commensurability	338
c. Tori and spheres	253	b. Growth rates in groups	340
d. Abelian fundamental groups	255	c. Preservation of growth rate	345
Lecture 25. Fundamental group of a bouquet of circles		Lecture 33. Polynomial and exponential growth	348
a. Covering of bouquets of circles	256	a. Dichotomy for linear orbits	348
b. Standard paths and elements of the free group	264	b. Natural questions	349

•	Contents
•	Contents

	c.	Growth rates in nilpotent groups	351
	d.	Milnor-Wolf Theorem	354
	Le	cture 34. Gromov's Theorem	356
	a.	General ideas	356
	b.	Large scale limit of two examples	359
	c.	General construction of a limiting space	364
	Lec	eture 35. Grigorchuk's group of intermediate growth	366
	a.	Automorphisms of binary trees	366
	b.	Superpolynomial growth	369
	c.	Subexponential growth	372
	Lec	eture 36. Coarse geometry and quasi-isometries	376
	a.	Coarse geometry	376
	b.	Groups as geometric objects	380
	c.	Finitely presented groups	382
	Lec	cture 37. Amenable and hyperbolic groups	385
	a.	Amenability	385
	b.	Conditions for amenability and non-amenability	-387
	c.	Hyperbolic spaces	390
	d.	Hyperbolic groups	393
	e.	The Gromov boundary	393
Η	ints	to selected exercises	395
S	ugge	stions for projects and further reading	401
Bibliography			409
Ir	ıdex		413