
Contents

Part I	1D Problems	1
1	1D Model Elliptic Problem	3
1.1	A Two-Point Boundary Value Problem	3
1.1.1	Classical Formulation	3
1.1.2	Interface Problem	4
1.1.3	Weak (Variational) Formulation of the Problem	5
1.2	Algebraic Structure of the Variational Formulation	7
1.3	Equivalence with a Minimization Problem	9
1.4	Sobolev Space $H^1(0, l)$	12
1.4.1	Distributional Derivatives	12
1.4.2	Finite Energy Solutions	14
1.4.3	Sobolev Space	14
1.5	Well Posedness of the Variational BVP	15
1.6	Examples from Mechanics and Physics	19
1.6.1	Elastic Bar	19
1.6.2	Heat Conduction	22
1.6.3	Vibrations of an Elastic Bar	24
1.6.4	Complex-Valued Problems	26
1.7	The Case with “Pure Neumann” BCs	27
	Exercises	29
2	Galerkin Method	33
2.1	Finite-Dimensional Approximation of the VBVP	33
2.1.1	Petrov–Galerkin Method	35
2.1.2	Internal Approximation	35
2.1.3	Orthogonality Relation	36
2.1.4	The Ritz Method	36
2.2	Elementary Convergence Analysis	36
2.2.1	SPD Case	37
2.2.2	A General Positive-Definite Case	38
2.3	Comments	39
	Exercises	39
3	1D hp Finite Element Method	41
3.1	1D hp Discretization	41

3.1.1	1D Master Element of Order p	42
3.1.2	Legendre Polynomials	43
3.1.3	1D Parametric Element of Arbitrary Order	44
3.1.4	1D hp Finite Element Space	45
3.2	Assembling Element Matrices into Global Matrices	46
3.3	Computing the Element Matrices	49
3.4	Accounting for the Dirichlet BC	51
3.5	Summary	52
3.6	Assignment 1: A Dry Run	53
	Exercises	54
4	1D hp Code	57
4.1	Setting Up the 1D hp Code	57
4.1.1	Makefile	58
4.1.2	Data Types	58
4.1.3	The Coding Protocol	58
4.2	Fundamentals	59
4.3	Graphics	60
4.3.1	Postscript Echo	64
4.4	Element Routine	65
4.5	Assignment 2: Writing Your Own Processor	66
4.5.1	Natural Order of Nodes and Nodal d.o.f.	67
	Exercises	69
5	Mesh Refinements in 1D	71
5.1	The h -Extension Operator. Constrained Approximation Coefficients	71
5.2	Projection-Based Interpolation in 1D	73
5.2.1	p Refinements	73
5.2.2	p Unrefinements	74
5.2.3	Implementing Projection-Based Interpolation	75
5.2.4	Initiating d.o.f. for p and h Unrefinements	76
5.3	Supporting Mesh Refinements	76
5.4	Data-Structure-Supporting Routines	77
5.4.1	Natural Order of Elements	78
5.4.2	Reconstructing Element Nodal Connectivities	78
5.5	Programming Bells and Whistles	79
5.5.1	Interfacing with the Frontal Solver	80
5.5.2	Adaptive Integration	81
5.5.3	Choice of Shape Functions	82
5.6	Interpolation Error Estimates	82
5.6.1	p -Interpolation Error Estimate	83
5.6.2	hp -Interpolation Error Estimate	85
5.7	Convergence	87
5.7.1	Uniform h Refinements	87
5.7.2	Uniform p Refinements	88

5.7.3	Adaptive h Refinements	88
5.7.4	Adaptive p Refinements	89
5.7.5	Adaptive hp Refinements	89
5.7.6	Adaptive p Refinements with an Optimal Initial Mesh	89
5.8	Assignment 3: Studying Convergence	90
5.8.1	A Coercive Elliptic Problem	90
5.8.2	A Convection-Dominated Diffusion Problem	90
5.9	<i>Definition of a Finite Element</i>	91
	Exercises	92
6	Automatic hp Adaptivity in 1D	95
6.1	The hp Algorithm	95
6.1.1	Projection-Based Interpolation in a Generalized Sense	96
6.1.2	The hp Algorithm	96
6.1.3	Unwanted h Refinements	100
6.1.4	Error Estimation. Stopping Criterion	101
6.1.5	Enforcing Optimal Refinements	102
6.2	Supporting the Optimal Mesh Selection	104
6.2.1	Determining the Optimal Refinement for an Element	105
6.3	Exponential Convergence. Comparing with h Adaptivity	107
6.3.1	The Shock Problem	107
6.3.2	The Problem with a Singular Solution	109
6.4	Discussion of the hp Algorithm	110
6.5	Algebraic Complexity and Reliability of the Algorithm	112
6.5.1	Cost of the Linear Solver	113
6.5.2	Cost of Mesh Optimization	113
6.5.3	Error Estimation	114
	Exercises	114
7	Wave Propagation Problems	117
7.1	<i>Convergence Analysis for Noncoercive Problems</i>	117
7.1.1	Interpretation with the Projection Operator	120
7.2	<i>Wave Propagation Problems</i>	121
7.2.1	Aubin–Nitsche Duality Argument	123
7.3	Asymptotic Optimality of the Galerkin Method	125
7.4	Dispersion Error Analysis	127
	Exercises	131
Part II	2D Elliptic Problems	133
8	2D Elliptic Boundary-Value Problem	135
8.1	Classical Formulation	135

8.1.1	Interface Problems	137
8.1.2	Regularity of the Solution	138
8.2	Variational (Weak) Formulation	139
8.2.1	Interface Problem	142
8.3	Algebraic Structure of the Variational Formulation	143
8.4	Equivalence with a Minimization Problem	144
8.5	Examples from Mechanics and Physics	145
8.5.1	The Membrane Problem	145
8.5.2	Torsion of a Shaft	146
8.5.3	Diffusion–Convection–Reaction Equation	146
	Exercises	146
9	Sobolev Spaces	149
9.1	Sobolev Space $H^1(\Omega)$	149
9.1.1	Distributional Derivatives	149
9.1.2	Finite-Energy Solutions. Sobolev Space of Order One	151
9.2	<i>Sobolev Spaces of an Arbitrary Order</i>	151
9.2.1	Sobolev Spaces of Arbitrary Integer Order	151
9.2.2	Sobolev Spaces of Arbitrary Real Order. Sloboditskiĭ’s Definition	153
9.2.3	Hörmander’s Definition	154
9.2.4	Interpolation Spaces	155
9.3	Density and Embedding Theorems	157
9.4	Trace Theorem	158
9.5	<i>Well Posedness of the Variational BVP</i>	159
9.5.1	Continuity of the Bilinear and Linear Forms	159
9.5.2	Coercivity	159
9.5.3	The Case with “Pure Neumann” Boundary Conditions	160
	Exercises	161
10	2D hp Finite Element Method on Regular Meshes	163
10.1	Quadrilateral Master Element	164
10.2	Triangular Master Element	166
10.3	Parametric Element	169
10.4	Finite Element Space. Construction of Basis Functions	169
10.4.1	Setting Up the Orientation for Edges	171
10.5	Calculation of Element Matrices	172
10.5.1	Computation of the Boundary Terms	174
10.5.2	Numerical Integration	175
10.6	Modified Element. Imposing Dirichlet Boundary Conditions	177
10.7	Postprocessing — Local Access to Element d.o.f.	179
10.8	Projection-Based Interpolation	180
	Exercises	183

11	2D <i>hp</i> Code	185
11.1	Getting Started	185
11.2	Data Structure in FORTRAN 90	186
11.3	Fundamentals	188
11.3.1	System Files, IO	188
11.3.2	The Main Program	188
11.3.3	Graphics	188
11.3.4	Quadrature Data	189
11.4	The Element Routine	189
11.5	Modified Element. Imposing Dirichlet Boundary Conditions	190
11.5.1	Imposing Dirichlet Boundary Conditions	191
11.6	Assignment 4: Assembly of Global Matrices	191
11.7	The Case with "Pure Neumann" Boundary Conditions	193
12	Geometric Modeling and Mesh Generation	195
12.1	Manifold Representation	195
12.2	Construction of Compatible Parametrizations	197
12.2.1	Transfinite Interpolation for a Rectangle	200
12.2.2	Transfinite Interpolation for a Triangle	201
12.3	Implicit Parametrization of a Rectangle	201
12.4	Input File Preparation	203
12.5	Initial Mesh Generation	205
12.5.1	The Case of "Pure" Neumann Conditions	209
12.5.2	Defining Profiles	210
13	The <i>hp</i> Finite Element Method on <i>h</i>-Refined Meshes	211
13.1	Introduction. The <i>h</i> Refinements	211
13.2	1-Irregular Mesh Refinement Algorithm	213
13.3	Data Structure in Fortran 90 (Continued)	217
13.3.1	The Genealogical Information for Nodes	217
13.3.2	The Natural Order of Elements	218
13.4	Constrained Approximation for C^0 Discretizations	219
13.4.1	Modified Element	220
13.5	Reconstructing Element Nodal Connectivities	222
13.5.1	Definition of the Modified Element	222
13.6	Determining Neighbors for Midedge Nodes	224
13.7	Additional Comments	225
13.7.1	Assembly of Global Matrices.	
	Interfacing with Solvers	225
13.7.2	Evaluation of Local d.o.f.	226
13.7.3	<i>p</i> Refinements	226
14	Automatic <i>hp</i> Adaptivity in 2D	227
14.1	The Main Idea	227

14.2	The 2D hp Algorithm	228
14.3	Example: L-Shape Domain Problem	234
14.4	Example: 2D “Shock” Problem	236
14.5	Additional Remarks	238
14.5.1	Error Computation in the Case of a Known Solution	238
14.5.2	Maximum Number of Iterations, Debugging	239
14.5.3	Systems of Equations and Complex-Valued Problems	239
14.5.4	Automatic h Adaptivity	239
15	Examples of Applications	241
15.1	A “Battery Problem”	241
15.2	Linear Elasticity	244
15.3	An Axisymmetric Maxwell Problem	251
	Exercises	256
16	Exterior Boundary-Value Problems	259
16.1	Variational Formulation. Infinite Element Discretization	260
16.1.1	System of Coordinates	261
16.1.2	Incorporating the Far-Field Pattern. Change of Dependent Variable	262
16.2	Selection of IE Radial Shape Functions	265
16.2.1	Same Trial and Test Shape Functions (Bubnov–Galerkin)	265
16.2.2	Different Trial and Test Shape Functions (Petrov–Galerkin)	266
16.3	Implementation	267
16.3.1	Data Structure, Interface with Frontal Solver	267
16.3.2	Choice of Radial Order N	268
16.3.3	Calculation of Infinite Element Stiffness Matrix. Routine <code>infel/eleminf</code>	269
16.4	Calculation of Echo Area	270
16.4.1	Direct Evaluation Using the IE Solution	270
16.4.2	Evaluation through Postprocessing	270
16.5	Numerical Experiments	271
16.5.1	Evaluation of the Approximation Error	271
16.5.2	Selection of Radial Shape Functions. Conditioning	271
16.5.3	Scattering of a Plane Wave on a Rigid Cylinder	273
16.5.4	Comparison of Bubnov–Galerkin and Petrov–Galerkin Formulations	275
16.5.5	Scattering of a Plane Wave on a Wedge	275
16.5.6	Does the Resolution of Singularities Affect the Quality of the Echo Area?	277

16.5.7	Evaluation of EA	277
16.6	Comments	279
	Exercises	280
Part III	2D Maxwell Problems	283
17	2D Maxwell Equations	285
17.1	Introduction to Maxwell's Equations.....	286
17.1.1	Wave Equation	288
17.1.2	Time-Harmonic Wave Equation.....	288
17.2	Variational Formulation	289
17.2.1	Nondimensionalization of Maxwell's Equations	294
	Exercises	296
18	Edge Elements and the de Rham Diagram	301
18.1	Exact Sequences	301
18.1.1	Nedelec's Triangular Element of the Second Type	303
18.1.2	Nedelec's Rectangular Element of the First Type	304
18.1.3	Nedelec's Triangular Elements of the First Type	305
18.1.4	Parametric Elements.....	308
18.2	Projection-Based Interpolation	309
18.3	De Rham Diagram	314
18.4	Shape Functions.....	316
18.4.1	Quadrilateral Element	316
18.4.2	Triangle of the Second Type	317
18.4.3	Triangle of the First Type	319
	Exercises	321
19	2D Maxwell Code	323
19.1	Directories. Data Structure	323
19.2	The Element Routine	325
19.3	Constrained Approximation. Modified Element	328
19.3.1	1D h -Extension Operator	328
19.3.2	2D h -Extension Operators	329
19.3.3	Adaptivity	330
19.3.4	Modified Element. Constrained Approximation	330
19.3.5	Interface with the Frontal Solver	331
19.4	Setting Up a Maxwell Problem	331
19.4.1	Example: Scattering of a Plane Wave on a Screen	332
	Exercises	334
20	hp Adaptivity for Maxwell Equations	337
20.1	Projection-Based Interpolation Revisited	338
20.1.1	Generalized Projection-Based Interpolation	339

20.1.2	Conflict between the Constrained Nodes and the Commutativity Property	340
20.2	The <i>hp</i> Mesh Optimization Algorithm	341
20.3	Example: The Screen Problem	345
21	Exterior Maxwell Boundary-Value Problems	351
21.1	Variational Formulation	352
21.2	Infinite Element Discretization in 3D.....	353
21.2.1	Infinite Element Coordinates	354
21.2.2	Incorporating the Far-Field Pattern	356
21.2.3	Discretization	357
21.3	Infinite Element Discretization in 2D.....	357
21.3.1	2D IE Coordinates	357
21.3.2	Incorporating the Far-Field Pattern	358
21.4	Stability	358
21.5	Implementation	360
21.5.1	Automatic <i>hp</i> Adaptivity	361
21.5.2	Choice of Radial Order N	361
21.5.3	Evaluation of the Error	362
21.6	Numerical Experiments	362
21.6.1	Scattering of a Plane Wave on a PEC Cylinder. Verification of the Code.....	362
21.6.2	Scattering of a Plane Wave on a PEC Wedge.....	364
21.6.3	Evaluation of RCS	365
	Exercises	367
22	A Quick Summary and Outlook	369
Appendix A	371
Bibliography	375
Index	389