Contents

Preface ix

About the Authors xiii

1 Introduction and Overview **1**

1.1 Lagrangian and Hamiltonian Formalisms 1
1.2 The Rigid Body 6
1.3 Lie–Poisson Brackets, Poisson Manifolds, Momentum Maps 9
1.4 The Heavy Top 16
1.5 Incompressible Fluids 18
1.6 The Maxwell–Vlasov System 22
1.7 Nonlinear Stability 29
1.8 Bifurcation 43
1.9 The Poincaré–Melnikov Method 47
1.10 Resonances, Geometric Phases, and Control 50

2 Hamiltonian Systems on Linear Symplectic Spaces **61**

2.1 Introduction 61
2.2 Symplectic Forms on Vector Spaces 66
2.3 Canonical Transformations, or Symplectic Maps 69
2.4 The General Hamilton Equations 74
2.5 When Are Equations Hamiltonian? 77
2.6 Hamiltonian Flows 80
2.7 Poisson Brackets 82

2.8 A Particle in a Rotating Hoop 87
2.9 The Poincaré–Melnikov Method 94

3 An Introduction to Infinite-Dimensional Systems 105
3.1 Lagrange's and Hamilton's Equations for Field Theory . . 105
3.2 Examples: Hamilton's Equations 107
3.3 Examples: Poisson Brackets and Conserved Quantities . . 115

4 Manifolds, Vector Fields, and Differential Forms 121
4.1 Manifolds 121
4.2 Differential Forms 129
4.3 The Lie Derivative 137
4.4 Stokes' Theorem 141

5 Hamiltonian Systems on Symplectic Manifolds 147
5.1 Symplectic Manifolds 147
5.2 Symplectic Transformations 150
5.3 Complex Structures and Kähler Manifolds 152
5.4 Hamiltonian Systems 157
5.5 Poisson Brackets on Symplectic Manifolds 160

6 Cotangent Bundles 165
6.1 The Linear Case 165
6.2 The Nonlinear Case 167
6.3 Cotangent Lifts 170
6.4 Lifts of Actions 173
6.5 Generating Functions 174
6.6 Fiber Translations and Magnetic Terms 176
6.7 A Particle in a Magnetic Field 178

7 Lagrangian Mechanics 181
7.1 Hamilton's Principle of Critical Action 181
7.2 The Legendre Transform 183
7.3 Euler–Lagrange Equations 185
7.4 Hyperregular Lagrangians and Hamiltonians 188
7.5 Geodesics 195
7.6 The Kaluza–Klein Approach to Charged Particles 200
7.7 Motion in a Potential Field 202
7.8 The Lagrange–d'Alembert Principle 205
7.9 The Hamilton–Jacobi Equation 210

8 Variational Principles, Constraints, & Rotating Systems 219
8.1 A Return to Variational Principles 219
8.2 The Geometry of Variational Principles 226
8.3 Constrained Systems 234

8.4 Constrained Motion in a Potential Field 238
8.5 Dirac Constraints 242
8.6 Centrifugal and Coriolis Forces 248
8.7 The Geometric Phase for a Particle in a Hoop 253
8.8 Moving Systems 257
8.9 Routh Reduction 260

9 An Introduction to Lie Groups **265**
9.1 Basic Definitions and Properties 267
9.2 Some Classical Lie Groups 283
9.3 Actions of Lie Groups 309

10 Poisson Manifolds **327**
10.1 The Definition of Poisson Manifolds 327
10.2 Hamiltonian Vector Fields and Casimir Functions 333
10.3 Properties of Hamiltonian Flows 338
10.4 The Poisson Tensor 340
10.5 Quotients of Poisson Manifolds 349
10.6 The Schouten Bracket 353
10.7 Generalities on Lie–Poisson Structures 360

11 Momentum Maps **365**
11.1 Canonical Actions and Their Infinitesimal Generators . . 365
11.2 Momentum Maps 367
11.3 An Algebraic Definition of the Momentum Map 370
11.4 Conservation of Momentum Maps 372
11.5 Equivariance of Momentum Maps 378

12 Computation and Properties of Momentum Maps **383**
12.1 Momentum Maps on Cotangent Bundles 383
12.2 Examples of Momentum Maps 389
12.3 Equivariance and Infinitesimal Equivariance 396
12.4 Equivariant Momentum Maps Are Poisson 403
12.5 Poisson Automorphisms 412
12.6 Momentum Maps and Casimir Functions 413

13 Lie–Poisson and Euler–Poincaré Reduction **417**
13.1 The Lie–Poisson Reduction Theorem 417
13.2 Proof of the Lie–Poisson Reduction Theorem for GL(n) . 420
13.3 Lie–Poisson Reduction Using Momentum Functions 421
13.4 Reduction and Reconstruction of Dynamics 423
13.5 The Euler–Poincaré Equations 432
13.6 The Lagrange–Poincaré Equations 442

14 Coadjoint Orbits **445**
14.1 Examples of Coadjoint Orbits 446
14.2 Tangent Vectors to Coadjoint Orbits 453
14.3 The Symplectic Structure on Coadjoint Orbits 455
14.4 The Orbit Bracket via Restriction of the Lie–Poisson Bracket 461
14.5 The Special Linear Group of the Plane 467
14.6 The Euclidean Group of the Plane 469
14.7 The Euclidean Group of Three-Space 474

15 The Free Rigid Body **483**
15.1 Material, Spatial, and Body Coordinates 483
15.2 The Lagrangian of the Free Rigid Body 485
15.3 The Lagrangian and Hamiltonian in Body Representation 487
15.4 Kinematics on Lie Groups 491
15.5 Poinsot's Theorem 492
15.6 Euler Angles 495
15.7 The Hamiltonian of the Free Rigid Body 497
15.8 The Analytical Solution of the Free Rigid-Body Problem . 500
15.9 Rigid-Body Stability 505
15.10 Heavy Top Stability 509
15.11 The Rigid Body and the Pendulum 514

References **521**

Index **555**