

Contents

Introduction	1
Advice for the Beginner	2
Information for the Expert	2
Prerequisites	6
Sources	6
Courses	7
A First Course	7
A Second Course	8
Acknowledgements	9
0 Elementary Definitions	11
0.1 Rings and Ideals	11
0.2 Unique Factorization	13
0.3 Modules	15
I Basic Constructions	19
1 Roots of Commutative Algebra	21
1.1 Number Theory	21
1.2 Algebraic Curves and Function Theory	23
1.3 Invariant Theory	24
1.4 The Basis Theorem	26
1.4.1 Finite Generation of Invariants	29

1.5	Graded Rings	29
1.6	Algebra and Geometry: The Nullstellensatz	31
1.7	Geometric Invariant Theory	37
1.8	Projective Varieties	39
1.9	Hilbert Functions and Polynomials	41
1.10	Free Resolutions and the Syzygy Theorem	44
1.11	Exercises	46
	Noetherian Rings and Modules	46
	An Analysis of Hilbert's Finiteness Argument	47
	Some Rings of Invariants	47
	Algebra and Geometry	49
	Graded Rings and Projective Geometry	51
	Hilbert Functions	53
	Free Resolutions	54
	Spec, max-Spec, and the Zariski Topology	54
2	Localization	57
2.1	Fractions	59
2.2	Horn and Tensor	62
2.3	The Construction of Primes	70
2.4	Rings and Modules of Finite Length	71
2.5	Products of Domains	78
2.6	Exercises	79
	Z-graded Rings and Their Localizations	81
	Partitions of Unity	83
	Gluing	84
	Constructing Primes	85
	Idempotents, Products, and Connected Components	85
3	Associated Primes and Primary Decomposition	87
3.1	Associated Primes	89
3.2	Prime Avoidance	90
3.3	Primary Decomposition	94
3.4	Primary Decomposition and Factoriality	98
3.5	Primary Decomposition in the Graded Case	99
3.6	Extracting Information from Primary Decomposition	100
3.7	Why Primary Decomposition Is Not Unique	102
3.8	Geometric Interpretation of Primary Decomposition	103
3.9	Symbolic Powers and Functions Vanishing to High Order	105
	3.9.1 A Determinantal Example	106
3.10	Exercises	108
	General Graded Primary Decomposition	109
	Primary Decomposition of Monomial Ideals	111
	The Question of Uniqueness	111
	Determinantal Ideals	112

Total Quotients	113
Prime Avoidance	113
4 Integral Dependence and the Nullstellensatz	117
4.1 The Cayley-Hamilton Theorem and Nakayama's Lemma	119
4.2 Normal Domains and the Normalization Process	125
4.3 Normalization in the Analytic Case	128
4.4 Primes in an Integral Extension	129
4.5 The Nullstellensatz	131
4.6 Exercises	135
Nakayama's Lemma	135
Projective Modules and Locally Free Modules	136
Integral Closure of Ideals	137
Normalization	137
Normalization and Convexity	138
Nullstellensatz	141
Three More Proofs of the Nullstellensatz	142
5 Filtrations and the Artin-Rees Lemma	145
5.1 Associated Graded Rings and Modules	146
5.2 The Blowup Algebra	148
5.3 The Krull Intersection Theorem	150
5.4 The Tangent Cone	151
5.5 Exercises	151
6 Flat Families	155
6.1 Elementary Examples	157
6.2 Introduction to Tor	159
6.3 Criteria for Flatness	161
6.4 The Local Criterion for Flatness	166
6.5 The Rees Algebra	170
6.6 Exercises	171
Flat Families of Graded Modules	175
Embedded First-Order Deformations	175
7 Completions and Hensel's Lemma	179
7.1 Examples and Definitions	179
7.2 The Utility of Completions	182
7.3 Lifting Idempotents	186
7.4 Cohen Structure Theory and Coefficient Fields	189
7.5 Basic Properties of Completion	192
7.6 Maps from Power Series Rings	198
7.7 Exercises	203
Modules Whose Completions Are Isomorphic	203
The Krull Topology and Cauchy Sequences	204
Completions from Power Series	205

Coefficient Fields	205
Other Versions of Hensel's Lemma	206
II Dimension Theory	211
8 Introduction to Dimension Theory	213
8.1 Axioms for Dimension	218
8.2 Other Characterizations of Dimension	220
8.2.1 Affine Rings and Noether Normalization	221
8.2.2 Systems of Parameters and Krull's Principal Ideal Theorem	222
8.2.3 The Degree of the Hilbert Polynomial	223
9 Fundamental Definitions of Dimension Theory	225
9.1 Dimension Zero	227
9.2 Exercises	228
10 The Principal Ideal Theorem and Systems of Parameters	231
10.1 Systems of Parameters and Parameter Ideals	234
10.2 Dimension of Base and Fiber	236
10.3 Regular Local Rings	240
10.4 Exercises	242
Determinantal Ideals	244
Hilbert Series of a Graded Module	245
11 Dimension and Codimension One	247
11.1 Discrete Valuation Rings	247
11.2 Normal Rings and Serre's Criterion	249
11.3 Invertible Modules	253
11.4 Unique Factorization of Codimension-One Ideals	256
11.5 Divisors and Multiplicities	259
11.6 Multiplicity of Principal Ideals	261
11.7 Exercises	264
Valuation Rings	264
The Grothendieck Ring	265
12 Dimension and Hilbert-Samuel Polynomials	271
12.1 Hilbert-Samuel Functions	272
12.2 Exercises	275
Analytic Spread and the Fiber of a Blowup	276
Multiplicities	276
Hilbert Series	280

13 The Dimension of Affine Rings	281
13.1 Noether Normalization	281
13.2 The Nullstellensatz	292
13.3 Finiteness of the Integral Closure	292
13.4 Exercises	296
Quotients by Finite Groups	296
Primes in Polynomial Rings	297
Dimension in the Graded Case	297
Noether Normalization in the Complete Case	298
Products and Reduction to the Diagonal	299
Equational Characterization of Systems of Parameters	301
14 Elimination Theory, Generic Freeness, and the Dimension of Fibers	303
14.1 Elimination Theory	303
14.2 Generic Freeness	307
14.3 The Dimension of Fibers	308
14.4 Exercises	314
Elimination Theory	314
15 Gröbner Bases	317
Constructive Module Theory	318
Elimination Theory	318
15.1 Monomials and Terms	319
15.1.1 Hilbert Function and Polynomial	320
15.1.2 Syzygies of Monomial Submodules	322
15.2 Monomial Orders	323
15.3 The Division Algorithm	330
15.4 Gröbner Bases	331
15.5 Syzygies	334
15.6 History of Gröbner Bases	337
15.7 A Property of Reverse Lexicographic Order	338
15.8 Gröbner Bases and Flat Families	342
15.9 Generic Initial Ideals	348
15.9.1 Existence of the Generic Initial Ideal	349
15.9.2 The Generic Initial Ideal is Borel-Fixed	351
15.9.3 The Nature of Borel-Fixed Ideals	352
15.10 Applications	355
15.10.1 Ideal Membership	355
15.10.2 Hilbert Function and Polynomial	355
15.10.3 Associated Graded Ring	356
15.10.4 Elimination	357
15.10.5 Projective Closure and Ideal at Infinity	359
15.10.6 Saturation	360

15.10.7 Lifting Homomorphisms	360
15.10.8 Syzygies and Constructive Module Theory	361
15.10.9 What's Left?	363
15.11 Exercises	365
15.12 Appendix: Some Computer Algebra Projects	375
Project 1. Zero-Dimensional Gorenstein Ideals	376
Project 2. Factoring Out a General Element from an sth Syzygy	377
Project 3. Resolutions over Hypersurfaces	377
Project 4. Rational Curves of Degree $r + 1$ in \mathbf{P}^r . . .	378
Project 5. Regularity of Rational Curves	378
Project 6. Some Monomial Curve Singularities	379
Project 7. Some Interesting Prime Ideals	379
16 Modules of Differentials	383
16.1 Computation of Differentials	387
16.2 Differentials and the Cotangent Bundle	388
16.3 Colimits and Localization	391
16.4 Tangent Vector Fields and Infinitesimal Morphisms . . .	396
16.5 Differentials and Field Extensions	397
16.6 Jacobian Criterion for Regularity	401
16.7 Smoothness and Generic Smoothness	404
16.8 Appendix: Another Construction of Kähler Differentials .	407
16.9 Exercises	409
III Homological Methods	417
17 Regular Sequences and the Koszul Complex	419
17.1 Koszul Complexes of Lengths 1 and 2	420
17.2 Koszul Complexes in General	423
17.3 Building the Koszul Complex from Parts	427
17.4 Duality and Homotopies	432
17.5 The Koszul Complex and the Cotangent Bundle of Projective Space	435
17.6 Exercises	437
Free Resolutions of Monomial Ideals	439
Conormal Sequence of a Complete Intersection	440
Regular Sequences Are Like Sequences of Variables . .	440
Blowup Algebra and Normal Cone of a Regular Sequence.	441
Geometric Contexts of the Koszul Complex	442

18 Depth, Codimension, and Cohen-Macaulay Rings	447
18.1 Depth	447
18.1.1 Depth and the Vanishing of Ext	449
18.2 Cohen-Macaulay Rings	451
18.3 Proving Primeness with Serre's Criterion	457
18.4 Flatness and Depth	460
18.5 Some Examples	462
18.6 Exercises	465
19 Homological Theory of Regular Local Rings	469
19.1 Projective Dimension and Minimal Resolutions	469
19.2 Global Dimension and the Syzygy Theorem	474
19.3 Depth and Projective Dimension: The Auslander-Buchsbaum Formula	475
19.4 Stably Free Modules and Factoriality of Regular Local Rings	480
19.5 Exercises	483
Regular Rings	484
Modules over a Dedekind Domain	484
The Auslander-Buchsbaum Formula	485
Projective Dimension and Cohen-Macaulay Rings .	485
Hilbert Function and Grothendieck Group	485
The Chern Polynomial	487
20 Free Resolutions and Fitting Invariants	489
20.1 The Uniqueness of Free Resolutions	490
20.2 Fitting Ideals	492
20.3 What Makes a Complex Exact?	496
20.4 The Hilbert-Burch Theorem	501
20.4.1 Cubic Surfaces and Sextuples of Points in the Plane	503
20.5 Castelnuovo-Mumford Regularity	504
20.5.1 Regularity and Hyperplane Sections	508
20.5.2 Regularity of Generic Initial Ideals	509
20.5.3 Historical Notes on Regularity	509
20.6 Exercises	510
Fitting Ideals and the Structure of Modules	510
Projectives of Constant Rank	513
Castelnuovo-Mumford Regularity	516
21 Duality, Canonical Modules, and Gorenstein Rings	519
21.1 Duality for Modules of Finite Length	520
21.2 Zero-Dimensional Gorenstein Rings	525
21.3 Canonical Modules and Gorenstein Rings in Higher Dimension	528

21.4 Maximal Cohen-Macaulay Modules	529
21.5 Modules of Finite Injective Dimension	530
21.6 Uniqueness and (Often) Existence	534
21.7 Localization and Completion of the Canonical Module	536
21.8 Complete Intersections and Other Gorenstein Rings	537
21.9 Duality for Maximal Cohen-Macaulay Modules	538
21.10 Linkage	539
21.11 Duality in the Graded Case	545
21.12 Exercises	546
The Zero-Dimensional Case and Duality	546
Higher Dimension	548
The Canonical Module as Ideal	551
Linkage and the Cayley-Bacharach Theorem	552
Appendix 1 Field Theory	555
A1.1 Transcendence Degree	555
A1.2 Separability	557
A1.3 p-Bases	559
A1.3.1 Exercises	562
Appendix 2 Multilinear Algebra	565
A2.1 Introduction	565
A2.2 Tensor Products	567
A2.3 Symmetric and Exterior Algebras	569
A2.3.1 Bases	572
A2.3.2 Exercises	574
A2.4 Coalgebra Structures and Divided Powers	575
A2.4.1 $S(M)^*$ and $S(M)$ as Modules over One Another	582
A2.5 Schur Functors	584
A2.5.1 Exercises	587
A2.6 Complexes Constructed by Multilinear Algebra	589
A2.6.1 Strands of the Koszul Complex	591
A2.6.2 Exercises	603
Appendix 3 Homological Algebra	611
A3.1 Introduction	611
Part I: Resolutions and Derived Functors	614
A3.2 Free and Projective Modules	615
A3.3 Free and Projective Resolutions	617
A3.4 Injective Modules and Resolutions	618
A3.4.1 Exercises	623
Injective Envelopes	623
Injective Modules over Noetherian Rings	623
A3.5 Basic Constructions with Complexes	626
A3.5.1 Notation and Definitions	626

A3.6 Maps and Homotopies of Complexes	627
A3.7 Exact Sequences of Complexes	631
A3.7.1 Exercises	632
A3.8 The Long Exact Sequence in Homology	632
A3.8.1 Exercises	634
Diagrams and Syzygies	634
A3.9 Derived Functors	636
A3.9.1 Exercise on Derived Functors	639
A3.10 Tor	639
A3.10.1 Exercises: Tor	639
A3.11 Ext	642
A3.11.1 Exercises: Ext	645
A3.11.2 Local Cohomology	649
Part II: From Mapping Cones to Spectral Sequences	650
A3.12 The Mapping Cone and Double Complexes	650
A3.12.1 Exercises: Mapping Cones and Double Complexes	654
A3.13 Spectral Sequences	656
A3.13.1 Mapping Cones Revisited	657
A3.13.2 Exact Couples	658
A3.13.3 Filtered Differential Modules and Complexes	661
A3.13.4 The Spectral Sequence of a Double Complex	665
A3.13.5 Exact Sequence of Terms of Low Degree	670
A3.13.6 Exercises on Spectral Sequences	671
A3.14 Derived Categories	677
A3.14.1 Step One: The Homotopy Category of Complexes	678
A3.14.2 Step Two: The Derived Category	679
A3.14.3 Exercises on the Derived Category	682
Appendix 4 A Sketch of Local Cohomology	683
A4.1 Local Cohomology and Global Cohomology	684
A4.2 Local Duality	686
A4.3 Depth and Dimension	686
Appendix 5 Category Theory	689
A5.1 Categories, Functors, and Natural Transformations	689
A5.2 Adjoint Functors	691
A5.2.1 Uniqueness	692
A5.2.2 Some Examples	692
A5.2.3 Another Characterization of Adjoints	693
A5.2.4 Adjoints and Limits	694
A5.3 Representable Functors and Yoneda's Lemma	695

Appendix 6 Limits and Colimits	697
A6.1 Colimits in the Category of Modules	700
A6.2 Flat Modules as Colimits of Free Modules	702
A6.3 Colimits in the Category of Commutative Algebras . . .	704
A6.4 Exercises	707
Appendix 7 Where Next?	709
Hints and Solutions for Selected' Exercises	711
References	745
Index of Notation	763
Index	767

*The selected exercises are marked with a *.