

CONTENTS

Part One

MODERN THEORIES OF DIFFERENTIATION AND INTEGRATION

CHAPTER I: DIFFERENTIATION

Lebesgue's Theorem on the Derivative of a Monotonic Function	3
1. Example of a Nondifferentiable Continuous Function	3
2. Lebesgue's Theorem on the Differentiation of a Monotonic Function. Sets of Measure Zero	5
3. Proof of Lebesgue's Theorem	6
4. Functions of Bounded Variation	9
Some Immediate Consequences of Lebesgue's Theorem	11
5. Fubini's Theorem on the Differentiation of Series with Monotonic Terms	11
6. Density Points of Linear Sets	12
7. Saltus Functions	13
8. Arbitrary Functions of Bounded Variation	15
9. The Denjoy-Young-Saks Theorem on the Derived Numbers of Arbitrary Functions	17
Interval Functions	19
10. Preliminaries	19
11. First Fundamental Theorem	21
12. Second Fundamental Theorem	22
13. The Darboux Integrals and the Riemann Integral	23
14. Darboux's Theorem	25
15. Functions of Bounded Variation and Rectification of Curves	26

CHAPTER II: THE LEBESGUE INTEGRAL

Definition and Fundamental Properties	29
16. The Integral for Step Functions. Two Lemmas	29
17. The Integral for Summable Functions	31
18. Term-by-Term Integration of an Increasing Sequence (Beppo Levi's Theorem)	33
19. Term-by-Term Integration of a Majorized Sequence (Lebesgue's Theorem)	36
20. Theorems Affirming the Integrability of a Limit Function	38
21. The Schwarz, Holder, and Minkowski Inequalities	40
22. Measurable Sets and Measurable Functions	43
 Indefinite Integrals. Absolutely Continuous Functions	 47
23. The Total Variation and the Derivative of the Indefinite Integral	47
24. Example of a Monotonic Continuous Function Whose Derivative Is Zero Almost Everywhere	48
25. Absolutely Continuous Functions. Canonical Decomposition of Monotonic Functions	50
26. Integration by Parts and Integration by Substitution	54
27. The Integral as a Set Function	56
 The Space L^2 and its Linear Functionals. L^p Spaces	 57
28. The Space L^2 ; Convergence in the Mean; the Riesz-Fischer Theorem	57
29. Weak Convergence	60
30. Linear Functionals	61
31. Sequence of Linear Functionals; a Theorem of Osgood	63
32. Separability of L^2 ; The Theorem of Choice	64
33. Orthonormal Systems	66
34. Subspaces of L^2 . The Decomposition Theorem	70
35. Another Proof of the Theorem of Choice. Extension of Functionals	72
36. The Space L^p and Its Linear Functionals	73
37. A Theorem on Mean Convergence	78
38. A Theorem of Banach and Saks	80
 Functions of Several Variables	 81
39. Definitions. Principle of Transition	81
40. Successive Integrations. Fubini's Theorem	83

41. The Derivative Over a Net of a Non-negative, Additive Rectangle Function. Parallel Displacement of the Net	84
42. Rectangle Functions of Bounded Variation. Conjugate Nets	87
43. Additive Set Functions. Sets Measurable (B)	89

Other Definitions of the Lebesgue Integral 92

44. Sets Measurable (L)	92
45. Functions Measurable (L) and the Integral (L)	94
46. Other Definitions. Egoroff's Theorem	96
47. Elementary Proof of the Theorems of Arzela and Osgood	100
48. The Lebesgue Integral Considered as the Inverse Operation of Differentiation	103

CHAPTER III: THE STIELTJES INTEGRAL AND ITS GENERALIZATIONS

Linear Functionals on the Space of Continuous Functions 105

49. The Stieltjes Integral	105
50. Linear Functionals on the Space C	106
51. Uniqueness of the Generating Function	111
52. Extension of a Linear Functional	112
53. The Approximation Theorem. Moment Problems	1 1 5
54. Integration by Parts. The Second Theorem of the Mean	118
55. Sequences of Functionals	119

Generalization of the Stieltjes Integral 122

56. The Riemann-Stieltjes and Lebesgue-Stieltjes Integrals	122
57. Reduction of the Lebesgue-Stieltjes Integral to That of Lebesgue	124
58. Relations Between Two Lebesgue-Stieltjes Integrals	126
59. Functions of Several Variables. Direct Definition	128
60. Definition by Means of the Principle of Transition	130

The Daniel Integral 132

61. Positive Linear Functionals	132
62. Functionals of Variable Sign	134
63. The Derivative of One Linear Functional With Respect to Another	137

*Part Two***INTEGRAL EQUATIONS. LINEAR TRANSFORMATIONS****CHAPTER IV: INTEGRAL EQUATIONS**

The Method of Successive Approximations	143
64. The Concept of an Integral Equation	143
65. Bounded Kernels	145
66. Square-Summable Kernels. Linear Transformations of the Space L^2	147
67. Inverse Transformations. Regular and Singular Values	151
68. Iterated Kernels. Resolvent Kernels	155
69. Approximation of an Arbitrary Kernel by Means of Kernels of Finite Rank	158
The Fredholm Alternative	161
70. Integral Equations With Kernels of Finite Rank	161
71. Integral Equations With Kernels of General Type	165
72. Decomposition Corresponding to a Singular Value	167
73. The Fredholm Alternative for General Kernels	170
Fredholm Determinants	172
74. The Method of Fredholm	172
75. Hadamard's Inequality	176
Another Method, Based on Complete Continuity	177
76. Complete Continuity	177
77. The Subspaces \mathfrak{R}_n and \mathfrak{R}_n'	179
78. The Cases $\nu = 0$ and $ \nu \geq 1$. The Decomposition Theorem	183
79. The Distribution of the Singular Values	187
80. The Canonical Decomposition Corresponding to a Singular Value	188
Applications to Potential Theory	190
81. The Dirichlet and Neumann Problems. Solution by Fredholm's Method	190

CHAPTER V: HILBERT AND BANACH SPACES

Hilbert Space	195
82. Hilbert Coordinate Space	195
83. Abstract Hilbert Space	197
84. Linear Transformations of Hilbert Space. Fundamental Concepts	200
85. Completely Continuous Linear Transformations	203
86. Biorthogonal Sequences. A Theorem of Paley and Wiener	208
Banach Spaces	210
87. Banach Spaces and Their Conjugate Spaces	210
88. Linear Transformations and Their Adjoints	215
89. Functional Equations	216
90. Transformations of the Space of Continuous Functions	219
91. A Return to Potential Theory	224

CHAPTER VI: COMPLETELY CONTINUOUS SYMMETRIC TRANSFORMATIONS OF HILBERT SPACE

Existence of Characteristic Elements. Theorem on Series Development	227
92. Characteristic Values and Characteristic Elements. Fundamental Properties of Symmetric Transformations	227
93. Completely Continuous Symmetric Transformations	231
94. Solution of the Functional Equation $j - \lambda Af = g$	235
95. Direct Determination of the n -th Characteristic Value of Given Sign	237
96. Another Method of Constructing Characteristic Values and Characteristic Elements	240
Transformations with Symmetric Kernel	242
97. Theorems of Hilbert and Schmidt	242
98. Mercer's Theorem	245
Applications to the Vibrating-String Problem and to Almost Periodic Functions	247
99. The Vibrating-String Problem. The Spaces D and H	247
100. The Vibrating-String Problem. Characteristic Vibrations	251
101. Space of Almost Periodic Functions	254

102. Proof of the Fundamental Theorem on Almost Periodic Functions	256
103. Isometric Transformations of a Finite-Dimensional Space	260

CHAPTER VII: BOUNDED SYMMETRIC, UNITARY, AND NORMAL TRANSFORMATIONS OF HILBERT SPACE

Symmetric Transformations	261
104. Some Fundamental Properties	261
105. Projections	266
106. Functions of a Bounded Symmetric Transformation	269
107. Spectral Decomposition of a Bounded Symmetric Transformation	272
108. Positive and Negative Parts of a Symmetric Transformation. Another Proof of the Spectral Decomposition	277
Unitary and Normal Transformations	280
109. Unitary Transformations	280
110. Normal Transformations. Factorizations	284
111. The Spectral Decomposition of Normal Transformations. Functions of Several Transformations	286
Unitary Transformations of the Space L^2	291
112. A Theorem of Bochner	291
113. Fourier-Plancherel and Watson Transformations	293

CHAPTER VIII: UNBOUNDED LINEAR TRANSFORMATIONS OF HILBERT SPACE

Generalization of the Concept of Linear Transformation	296
114. A Theorem of Hellinger and Toeplitz. Extension of the Concept of Linear Transformation	296
115. Adjoint Transformations	299
116. Permutability. Reduction	301
117. The Graph of a Transformation	303
118. The Transformation $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$	307
Self-Adjoint Transformations. Spectral Decomposition	308
1 19. Symmetric and Self-Adjoined Transformations. Definitions and Examples	308

120. Spectral Decomposition of a Self-Adjoint Transformation	313
121. Von Neumann's Method. Cayley Transforms	320
122. Semi-Bounded Self-Adjoint Transformations	323

Extensions of Symmetric Transformations 325

123. Cayley Transforms. Deficiency Indices	325
124. Semi-Bounded Symmetric Transformations. The Method of Friedrichs	329
125. Krein's Method	336

CHAPTER IX: SELF-ADJOINT TRANSFORMATIONS. FUNCTIONAL CALCULUS, SPECTRUM, PERTURBATIONS

Functional Calculus 341

126. Bounded Functions	341
127. Unbounded Functions. Definitions	343
128. Unbounded Functions. Rules of Calculation	346
129. Characteristic Properties of Functions of a Self-Adjoint Transformation	351
130. Finite or Denumerable Sets of Permutable Self-Adjoint Transformations	355
131. Arbitrary Sets of Permutable Self-Adjoint Transformations	358

The Spectrum of a Self-Adjoint Transformation and Its Perturbations 360

132. The Spectrum of a Self-Adjoint Transformation. Decomposition in Terms of the Point Spectrum and the Continuous Spectrum	360
133. Limit Points of the Spectrum	363
134. Perturbation of the Spectrum by the Addition of a Completely Continuous Transformation	367
135. Continuous Perturbations	368
136. Analytic Perturbations	373

CHAPTER X: GROUPS AND SEMIGROUPS OF TRANSFORMATIONS

Unitary Transformations 380

137. Stone's Theorem	380
138. Another Proof. Based on a Theorem of Bochner	385

139. Some Applications of Stone's Theorem	388
140. Unitary Representations of More General Groups	391
Non-Unitary Transformations	393
141. Groups and Semigroups of Self-Adjoint Transformations	393
142. Infinitesimal Transformation of a Semigroup of Transformations of General Type	397
143. Exponential Formulas	399
Ergodic Theorems	406
144. Fundamental Methods	406
145. Methods Based on Convexity Arguments	410
146. Semigroups of Nonpermutable Contractions	412
CHAPTER XI: SPECTRAL THEORIES FOR LINEAR TRANSFORMATIONS OF GENERAL TYPE	
Applications of Methods from the Theory of Functions	415
147. The Spectrum. Curvilinear Integrals	415
148. Decomposition Theorem	418
149. Relations Between the Spectrum and the Norms of Iterated Transformations	423
150. Application to Absolutely Convergent Trigonometric Series	427
151. Elements of a Functional Calculus	431
152. Two Examples	434
Von Neumann's Theory of Spectral Sets	435
153. Principal Theorems	435
154. Spectral Sets	439
155. Characterization of Symmetric, Unitary, and Normal Transformations by Their Spectral Sets	443
Bibliography	447
Appendix	
Extensions of Linear Transformations in Hilbert Space Which Extend Beyond This Space	457
Index	493
Notation & Symbols	504